B. THE QUASI-AUTONOMOUS PERIODIC PROBLEM

I. The Linear Case: Hilbertian Theory and Applications

In this paragraph, $H$ is a complex Hilbert space and $-A: D(A) \subset H \to H$ the generator of $S(t)$, a continuous semi-group of linear operators in $H$.

Given $f \in L^1(0,T;H)$ ($T > 0$), we are concerned with the problem of existence of $T$-periodic solutions of the equation:

\[
(1) \quad u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds.
\]

It is well-known, even for $H = \mathbb{C}$, that this problem need not have always a solution since the resonance phenomenon can occur for special $f$.

In case $H = \mathbb{R}^N$, it is usual to look for existence conditions by introducing the "adjoint" equation. Here we present a treatment in which the adjoint operator $A^*$ appears only at the very end as a technical tool to write down the conditions. The advantage of this method is to apply without change to "good" infinite dimensional cases, and give a general strategy for the bad ones (typically $A^* = -A$ and $A$ unbounded).

Lecture 19: Some general results

It is convenient to introduce the pulsation

\[
(2) \quad \omega = \frac{2\pi}{T}.
\]
Also, for $f \in L^1(0,T;H)$ arbitrary, and $m \in \mathbb{Z}$, we set

$$\hat{f}_m = \frac{1}{T} \int_0^T f(t) e^{-im\omega t} dt$$

With these notations, we can now state

**Theorem 1.** Let $u$ be a solution of (1) such that $u(0) = u(T)$. Then

$$\forall m \in \mathbb{Z}, \quad \hat{f}_m = (A + im\omega) \hat{u}_m.$$ 

**Corollary 2.** If there exists $u_0 \neq 0$ such that $S(T)u_0 = u_0$, some integer multiple of $i\omega$ is an eigenvalue for $A$.

**Corollary 3.** If (1) has a solution such that $u(0) = u(T)$, we have necessarily

$$\forall m \in \mathbb{Z}, \quad \hat{f}_m \in R(A + im\omega I).$$

**Proof of Theorem 1.** Let $h > 0$ and $t \in [0,T-h]$. We have:

$$u(t+h) = S(h)u(t) + \int_0^t S(t+h-s)f(s)ds$$

Dividing through by $h$, we obtain:

$$\frac{u(t+h)-u(t)}{h} = \frac{1}{h} \int_t^{t+h} S(t+h-s)f(s)ds + \frac{S(h)-I}{h} u(t).$$