In [3] Verbeek gives a general definition of a semigroup extension, which covers Schreier - and ideal - extensions as special cases (Definition 1, p. 22), as follows:

Definition 1. Let $A, S$ be semigroups. The pair $(E, \delta)$, where $\delta$ is a congruence on $E$, is a semigroup extension of $A$ by $S$, iff $E/\delta \cong S$ and there is a subsemigroup $A'$ of $E$, isomorphic to $A$, which is a $\delta$-class.

According to this definition, Verbeek shows that there exist extensions of $A$ by $S$ if and only if $S$ contains an idempotent element. (Theorem 1, p. 23). So for finite $S$ there is always an extension of arbitrary $A$ by $S$. Restricting this general notion of semigroup extension, the concept of union extension is defined by Verbeek as follows:

Definition 2. Let $A, S$ be semigroups and $(E, \delta)$ a semigroup extension. The pair $(E, \delta)$ is a union-extension of $A$ by $S$, iff the restriction of $\delta$ to $E\setminus A'$ is the identity relation, ($A'$ is the subsemigroup of definition 1).

Note that, according to definition 2, an extension $(E, \delta)$ of $A$ of $S$ with $0$ as extension idempotent is a union-extension iff $E$ is an ideal extension of $A$ by $S$.

There is a constructive method to obtain the set of all union-extensions (up to isomorphisms) of $A$ by $S$ for finite $A$ and $S$, as is indicated in the following theorem, due to Verbeek (Theorem 11, p. 40):

Theorem 1. Let $A, S$ be disjoint semigroups, $i \in S$ an idempotent element. For $E = A \cup S^-$, where $S^- = S \setminus \{i\}$, define an associative multiplication $o$ such that the following conditions hold for all $a, b \in A, s, t \in S^-$: $a \circ b = ab$

\[
\begin{align*}
a \circ s &= \begin{cases} 
  is & \text{if } is \neq i \\
  E & \text{if } is = i 
\end{cases} \\
s \circ a &= \begin{cases} 
  si & \text{if } si \neq i \\
  E & \text{if } si = i 
\end{cases} \\
s \circ t &= \begin{cases} 
  st & \text{if } st \neq i \\
  E & \text{if } st = i 
\end{cases} 
\end{align*}
\]
Then \( ((E, o), o) \) is a union-extension of \( A \) by \( S \) for \( \delta = A \times A \cup \{(x, x) \mid x \in S^-\} \).

Moreover, any union-extension \( (E', \delta') \) of \( A \) by \( S \) is isomorphic to one constructed in this way, where \( i \) is the extension idempotent, (cf. Theorem 1, [1]).

The following question is raised now: what conditions on \( A \) and/or \( S \) determine the existence of union extensions of \( A \) by \( S \) and a construction of all union extensions of \( A \) by \( S \). This problem is attacked and partly solved by Verbeek for special compositions of \( S \).

**Definition 3.** Let \( S \) be a semigroup containing the idempotent element \( i \) and let \( S^- = S \setminus i \). Define the subsets 1) through 9) of \( S^- \) by

1) \( U^- = \{s \in S^- : is = s, si = s\} \)
2) \( W^- = \{s \in S^- : is = s, si \neq s, si \neq i\} \)
3) \( V^- = \{s \in S^- : is = s, si \neq s, si \neq i\} \)
4) \( W' = \{s \in S^- : is = i, si = s\} \)
5) \( V' = \{s \in S^- : is = i, si = i\} \)
6) \( Z = \{s \in S^- : is = i, si \neq s, si \neq i\} \)
7) \( Y = \{s \in S^- : is \neq s, is = i, si = s\} \)
8) \( X = \{s \in S^- : is \neq s, is \neq i, si = i\} \)
9) \( X = \{s \in S^- : is \neq s, is \neq i, si = i\} \).

A composition of \( S \) with respect to \( i \) is a union of the non-empty subsets 1) through 9) of \( S^- \), which are contained in \( S^- \). For some compositions of \( S \) Verbeek derives necessary and sufficient conditions on \( A \) for the existence of union extensions of \( A \) by \( S \) ([1], Chapter 4). This was extended by Bröck and Jürgensen with the aid of a computer ([1], Tables 1 and 2).

The set of all possible compositions of semigroups was described partly by Verbeek ([3] and Chapter 4), and fully by van Leeuwen ([2]). Since the work of van Leeuwen ([2]) is not generally available it seems appropriate to describe the set of all possible compositions explicitly.

For all 130 combinations of non-empty subsets from Definition 3, 1) through 9), we show that they are possible compositions by giving examples of semigroups \( S \) whose composition consists of 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 non-empty subsets. We show also that the examples can be chosen in such a way that they are bands. It is hard to find a general method for constructing such bands and, in fact, we tried to construct these bands "inductively", using bands \( S \), which can be subsemigroups of the band one wants to construct.

For the composition \( S^- = U^- \cup Y \cup X \) (34) one can show that it is impossible