Remark on bifurcation problems with several parameters
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Abstract Some ideas on bifurcation problems with several parameters are collected and different phenomena are illustrated on examples of bifurcation problems which one can solve explicitly. The examples are dealing not only with differential equations but also with other fields.

1. Introduction
Bifurcation phenomena with several parameters have got growing interest recently because they occur in many applications. One is studying often in science the influence of several sources, for instance a beam under torsion and pressure, and it is quite natural to consider differential equations with several parameters.

The phenomena even with only one parameter can be rather complicated (compare the selections in Collatz [76], [77], and the variety of phenomena is increasing strongly with the number of parameters. Here only some ideas for multiparametric bifurcation problems are mentioned and some examples are added, in which one can solve the bifurcation problem explicitly and in which one can be sure to have got all branches. Bifurcation problems occur in many different areas of mathematics and applications and therefore we will not restrict ourselves to differential equations (D.E.). But D.E. are probably the first area, in which multiparametric bifurcation problems were considered in more detail; (see f.i. Stakgold [71], Atkinson [72] Dickey [77] Sleman [74], [79] a.o.) especially the linear case (multiparametric eigenvalue problems) has been studied (see for instance Gut [66], Hadeler [67], Collatz [68] Browne-Sleeman [80], a.o.).

2. Bifurcation sets of different order
Let M be a set of elements u, v, ... and \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \) a real-valued vector. We consider a given "relation" between certain element u and certain \( \lambda \), which we write in form of an equation \( F(u, \lambda) = 0 \). A pair \((u, \lambda)\) with \( F(u, \lambda) = 0 \) is called a "solution". \( N \) may be the set of all pairs \((u, \lambda)\) and \( S \) the set of all solutions.
We make the following assumptions.

1. For certain "smooth" subsets \( Q \) of \( N \) there is defined a "dimension" \( d \) as integer. Often one can define an analytic manifold as "smooth". \( N \) may have the dimension \( m \).

2. The set \( S \) can be represented as \( S = \bigcup_{j=1}^{p} S_j \); \( p = \infty \) is admitted; the subsets \( S_j \) are smooth and have the dimension \( d_j > 0 \); \( S_j \) is called a "branch" or bifurcation set of order \( m-d_j \).

3. The intersection \( S_{jk} \) of two branches \( S_j, S_k \) has a smaller dimension \( d_{jk} \) as each of the branches \( S_j, S_k \): \( (2.1) \quad d_{jk} = \dim S_{jk} = \dim (S_j \cap S_k) < \min (d_j, d_k) \) for \( j \neq k \).

We call also \( S_{jk} \) a "bifurcation set" of order \( m-d_{jk} \).

\( c_\ell \) may be the number of bifurcation sets order \( \ell \). Then we give the set \( S \) the symbol \( \left[ \begin{array}{cccc} c_1 & c_2 & c_3 & \cdots & c_m \end{array} \right] \); some or all of the constants \( c_\ell \) may be infinity.

The ideas of Hypergraphs (Berge [73]) Connectivity, chains, circles a.o.) can be used for the bifurcation set \( S \). [In Collatz [77] the row of the \( c_\ell \) is opposite and there \( \left[ c_2 \ c_1 \right] \) is written instead of \( \left[ c_1 \ c_2 \right] \) because for more parameters sometime \( c_1 \) is more important than \( c_2, \ldots \)].

3. Different Formulations of bifurcation problems

Different formulations can cause different bifurcation diagrams.

A. Different geometrical interpretation

a) We consider three curves \( G_1, G_2, G \) in the \( x-y \)-plane. Let \( \lambda \) be the arc-length on \( G \) and \( P(\lambda) \) the point of \( G \) belonging to \( \lambda \), furthermore \( r_j(\lambda) \) may be the length of a lot from \( P(\lambda) \) to the curve \( G_j \) \((j=1,2)\), Fig.1; \( r_j(\lambda) \) can be multivalued. The set of all graphs of \( r_j(\lambda) \) in a \( \lambda-r \)-plane is the bifurcation diagram. Fig.2 shows as example a case in which all three curves \( G_1, G_2, G \) are straight lines \( g_1, g_2, g \); the diagram of Fig.4 contains a cutting point \( Q^* \) (with coordinate \( \lambda^* \)) of the lines \( r_1(\lambda), r_2(\lambda) \), but this point \( Q^* \) is not a bifurcation point, because one can not go continuously from \( r_1(\lambda^*) \) to \( r_2(\lambda^*) \).

b) We take circles \( C_j \) of radius \( r_j(\lambda) \) with \( P(\lambda) \) as center, which are touching the curve \( G_j \) \((j=1,2)\). The bifurcation diagram is the same as in a), but the cutting points are now bifurcation points, because there is a continuous transition between the correspon-