Norm inequalities for derivatives

1. Introduction. We consider the inequality

\[ \|y^{(k)}\|_{q,w} \leq K \|y\|_{p,w}^\alpha \|y^{(n)}\|_{r,w}^\beta \]  

(1.1)

with norms given by

\[ \|y\|_{p,w}^p = \int_J |y(t)|^p w(t) \, dt \text{ when } 1 \leq p < \infty \]

and

\[ \|y\|_{p,w} = \text{ess. sup } |y(t)| w(t) \text{ when } p = \infty, t \in J \]

where \( w \) is a nonnegative locally integrable function. In place of \( \|\cdot\|_{p,1} \)
we also use \( \|\cdot\|_{p,w} \). Throughout this paper \( J \) will denote the whole line \( \mathbb{R} \) or the half-line \( \mathbb{R}^+ = (0, \infty) \) unless the contrary is explicitly stated.

In section 2 we discuss the classical case when \( w(t) = 1 \) a.e. In addition to the known results for this case we will survey some new ones which have not yet been published. Section 3 deals with the case when there is a non unit weight function. Most of the results here are new. Detailed proofs will be published elsewhere. Proofs will be outlined here in those instances where meaningful outlines can be given without exceeding our space allotment.

By \( W_{p,r}^n (w) \) we denote the set of all complex valued functions \( y \) in \( L_{w}^p (J) \) such that \( y^{(n-1)} \) is absolutely continuous on all compact subintervals of \( J \) and \( y^{(n)} \) is in \( L_{w}^r (J) \). The set \( W_{p,r}^n (1) \) is also denoted by \( W_{p,r}^n \) and \( W_{p,r}^n (w) \) denotes \( W_{p,r}^n (w) \) when \( p = r \).

2. The case \( w(t) = 1 \) a.e. The values of the parameters for which (1.1) holds are known.

Theorem 1. (Gabushin [29]). Let \( J = \mathbb{R} \) or \( \mathbb{R}^+ \). Let \( n, k \) be integers satisfying \( 1 \leq k < n \). Let \( 1 \leq p, r \leq \infty \). There is a finite constant \( K \) such that
\[ ||y^{(k)}||_q \leq K||y||_p^\alpha ||y^{(n)}||_r^\beta \]  \hspace{1cm} (2.1)

holds for all \( y \) in \( W^n_{p,r} \) if and only if
\[ nq^{-1} \leq (n - k)p^{-1} + kr^{-1} \]  \hspace{1cm} (2.2)

and
\[ \alpha = (n - k - r^{-1} + q^{-1})/(n - r^{-1} + p^{-1}), \quad \beta = 1 - \alpha. \]  \hspace{1cm} (2.3)

Here we are following the usual convention that \( s^{-1} = 0 \) when \( s = \infty \).

Given \( p, r \) with \( 1 \leq p, r \leq \infty \) and integers \( n, k \) with \( 1 \leq k < n \) if \( q \) satisfies (2.2) then inequality (1.1) holds for some finite constant \( K \) provided that \( \alpha \) and \( \beta \) are determined by (2.3). Clearly if (1.1) holds for some constant \( K \) then there is a smallest such constant. This best constant depends on \( n, k, p, q, r \) and \( J \) and so we will denote it by \( K = K(n, k, p, q, r; J) \). In case \( p = q = r \) this constant will also be denoted by \( K(n, k, p; J) \). The value of \( K \) is not affected if the interval \( J = R^+ = (0, \infty) \) is replaced by any interval of the form \( (a, \infty) \) or \((-\infty, a), -\infty < a < \infty\), as can be seen from the simple transformations \( t \mapsto t + a \) and \( t \mapsto -t \).

The table below summarizes the known values of \( K \) along with the author and reference.