12. Spectral decomposition of $\mathbb{E}L^2(1\backslash \mathbb{C}, \chi)$.

12.1. Introduction.

Now we have got the analytical continuation of the Eisenstein model, we may derive the spectral decomposition of $\mathbb{E}L^2$. The points $s_0 \in (0, \frac{1}{2}(1-\tau)]$ at which $E(s)$ has a pole give non-cuspidal invariant subspaces of $\mathbb{E}L^2$ generated by the $E \in E(s)$ with $s \in E(s_0)$. The orthogonal complement in $\mathbb{E}L^2$ of these subspaces is given by a direct integral.

For functions of the form $*\Theta*\phi\xi$ with $\xi$ a holomorphic rapidly decreasing section of $*H^T$ I give a decomposition of the scalar product in 12.2. From this decomposition it becomes clear how $\mathbb{E}L^2$ is built up. It has some closed invariant subspaces, the orthogonal complement $\mathbb{L}^2$ is a continuous integral of representation of the principal series. This space $\mathbb{L}^2$ I discuss in 12.3. We need a scalar product formula for more general functions than considered in 12.2; this extension is given in 12.5. The Eisenstein transform occurring in this formula is introduced in 12.4. The results of this section are reformations in our notations of know results; compare [37], Satz 11.2 and Satz 12.3.

12.2. Integral formulas

Let $\sigma > \frac{1}{2}$. From proposition 11.4.2 we know that for holomorphic rapidly decreasing sections $\xi$ and $\eta$ of $*H^T$ on $|\text{Re } s| \leq \sigma$

$$\left(\mathcal{O}*\phi\xi, \mathcal{O}*\phi\eta\right) = \frac{1}{2\pi i} \int_{\text{Re } s = -\sigma} \{(1+M)\pi_1 \xi, \pi_1 \eta\}(s) ds. \tag{12.2.1}$$

On the line $\text{Re } s = 0$ the operator $M(s)$ is unitary, hence bounded; lemma 11.6.5 gives boundedness on the strip $0 < \text{Re } s \leq \sigma$. This implies that $\{(1+M)\pi_1 \xi, \pi_1 \eta\}(s)$ is rapidly decreasing for $|\text{Im } s| + \infty$ on the strip $-\sigma < \text{Re } s \leq 0$ outside a neighbourhood of the poles. We can change the path of integration to the line $\text{Re } s = 0$. Doing this there occur residues at the points in the following set:

12.2.1. Definition. $S^\sigma = \{s_0 \in (0, \frac{1}{2}(1-\tau)] | E(s_0) \neq \{0\}\}$.

On the line $\text{Re } s = 0$ I rewrite

$$\{(1+M)\pi_1 \xi, \pi_1 \eta\}(s) = \frac{1}{2}\{(1+M)^2\pi_1 \xi, \pi_1 \eta\}(s) \quad \text{(see proposition 9.4.4)}$$

$$= \frac{1}{2}\{(1+M)\pi_1 \xi, \pi_1 \eta\}(s) + \frac{1}{2}\{(1+M)\pi_1 \xi, \pi_1 \eta\}(-s) \quad \text{(see (3.5.17))}. \tag{12.2.2}$$

So we get

$$\frac{1}{2\pi i} \int_{\text{Re } s = 0} \frac{1}{2}\{(1+M)\pi_1 \xi, \pi_1 \eta\}(s) ds + \frac{1}{2\pi i} \int_{\text{Re } s = 0} \frac{1}{2}\{(1+M)\pi_1 \xi, \pi_1 \eta\}(s) ds$$

$$= \frac{1}{4\pi i} \int_{\text{Re } s = 0} \{(1+M)\pi_1 \xi, (1+M)\pi_1 \eta\}(s) ds.$$
The integrand is invariant under \( s \to -s \), so we may integrate over \((0,i^\infty)\) = \( \{it \mid t > 0\} \). We get the following result:

12.2.2. Lemma. Let \( \sigma > 1/2 \). For holomorphic rapidly decreasing sections \( \xi \) and \( \eta \) of \( *H^1 \) on \( |Re s| \leq \sigma \):

\[
(12.2.3) \quad \langle \*s H_0 \xi, \*s H_0 \eta \rangle = \left\{ \frac{1}{2 \pi i} \int_0^{i^\infty} \left\{ (1+iM) \pi_1 \xi(s), (1+iM) \pi_1 \eta(s) \right\} ds + \sum_{s_0 \in \mathbb{E}} \left\{ \text{res} \ \tau(s) \pi_1 \xi(s_0), \pi_1 \eta(s_0) \right\} s_0 \right\}.
\]

The \( \*s H \xi \) form a dense set in \( eL^2 \). Let \( s_0 \in \mathbb{S}_0 \), \( e \in \mathcal{E}(s_0) \) and \( \phi(s_0) \in \pi_1 H^\dagger(s_0) \). Then \( E_{s_0} \phi(s_0) \in eL^2 \), see proposition 11.7.11. It is not difficult to approximate \( \phi(s_0) \) by elements of the form \( \*s H \xi \):

\[
(12.2.4) \quad \xi_{\nu}(s) = e^{-\nu(s-s_0)^2} \phi(s) \omega_\xi.
\]

For \( \nu > 0 \) this is a holomorphic rapidly decreasing section of \( \pi_1 H^\dagger \) on any strip \( |Re s| \leq \sigma \) and

\[
(12.2.5) \quad \lim_{\nu \to 0} \*s H \xi_{\nu} = E_{s_0} \phi(s_0) \text{ in } eL^2(\mathbb{N}^\dagger, \chi).
\]

Proof. It is clear that for any holomorphic rapidly decreasing section \( \xi \) of \( \pi_1 H^\dagger \)

\[
(12.2.6) \quad \lim_{\nu \to 0} \langle \*s H \xi, \*s H \xi_{\nu} \rangle = \left\{ \text{res} \ \tau(s) \pi_1 \xi(s_0), (s_0) \omega_\xi - s_0 \right\}.
\]

Suppose in addition that \( \*s H \xi \in \mathcal{S}_\infty^e \), see 11.4.1. The \( \*s H \xi \) of this form already are dense in \( eL^2 \). Now \( \*s H \xi \in \mathcal{C}_\infty^e(\mathbb{N}^\dagger, \chi) \), so we get

\[
(12.2.7) \quad \langle \*s H \xi, E_{s_0} \phi(s_0) \rangle = \lim_{s \to s_0} \left( P^* \xi, P^* \phi(s) \omega_\xi \right) \text{ (see propositions 8.3.3 and 11.7.11)}
\]

\[
= \lim_{s \to s_0} \left( \pi_1(\xi(s), (s) M(s) \phi(s) \omega_\xi \right) \text{ (see propositions 11.3.1 and 11.3.3)}
\]

\[
= \lim_{s \to s_0} \left( \text{res} \ \tau(s) \pi_1 \xi(s_0), \pi_1(\xi(s), \phi(s) \omega_\xi \right) \text{ s_0}
\]

Comparison of (12.2.6) and (12.2.7) proves the lemma.

12.2.4. Definition. Let \( s_0 \in \mathbb{S}_e \). Define

\[
(12.2.8) \quad eL^2(s_0) = \{ E_{s_0} \phi(s_0) \mid \phi \in \mathcal{E}(s_0), \phi(s_0) \in \pi_1 H^\dagger(s_0) \}
\]

This is a noncuspidal invariant subspace of \( L^2 \).

12.2.5. Lemma. Let \( s_0 \in \mathbb{S}_e \), let \( \xi \) and \( \eta \) be as in lemma 12.2.2 and let \( \*s H \xi \) and \( \*s H \eta \) be the projections of \( \*s H \xi \) and \( \*s H \eta \) in \( eL^2(s_0) \). Then

\[
(12.2.9) \quad \langle \*s H \xi, \*s H \eta \rangle = \left\{ \text{res} \ \tau(s) \pi_1 \xi(s_0), \pi_1 \eta(s_0) \right\} s_0.
\]