A HIERARCHY OF CUTS IN MODELS OF ARITHMETIC

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Abstract In this paper we show that it is possible to classify most of the natural families of cuts considered to date in terms of a single hierarchy. This classification gives conservation and independence results for fragments of arithmetic.

Acknowledgment Over the past three years I have been in frequent and completely open communication with George Mills and Laurie Kirby concerning the problems discussed in this paper. Obviously I wish to express my sincere thanks to them for their considerable help with ideas and suggestions.

I also wish to thank Peter Aczel for generously providing a key lemma in the proof of the main theorem.

Preliminaries Let P be Peano's 1st order axioms for arithmetic and let $I\Sigma_n$ be the same set of axioms but with the induction shema restricted to Σ_n formulas. Throughout this paper M is a countable non-standard model of $I\Sigma_1$ and N is the standard model of P which, as usual, we identify with an initial segment of M. As far as possible we use n, m, k for elements of N, a, b, c, d, e for elements of M and $\alpha, \beta, \gamma, \delta$ for ordinals. For $a \in M$ we identify a with $\{x | x \in M \ \&\ x < a\}$, so a is also a subset of M.

Let $A \subseteq M$, $a \in M$. We write $A \gtrless a$ if every element of A is greater than (resp. less than) a. $I \subseteq M$ is a cut in M, denoted $I \subseteq_e M$, if $2 \in I$, I is closed under multiplication, $I \neq M$, and $a < b \in I \Rightarrow a \in I$.

Let $I \subseteq_e M$. If $A \subseteq I$ we say A is coded in M if there is some $b \in M$ such that for $a \in I$.

$$a \in A \Longleftrightarrow \text{the } a\text{'th prime divides } b.$$

Similiarly we can talk of $f: I \to M$ being coded etc.
Henceforth all subsets of I or functions on I etc. which are mentioned are assumed to be coded in M unless otherwise stated.

Suppose $f: I \to a$ is coded by e in M. Then by overspill there is a $b > I$ such that e codes a map $f': b \to a$ with $f' \upharpoonright I = f$. Hence we have a continuation of f a little way above I. This justifies the use of expressions like $f(c)$ for c close to, but exceeding, I.

Let X, V be sets of cuts in M. We say that X and V are symbiotic if for all $a, b \in M$,

$$\exists J \in X,\ a \in J < b \Longleftrightarrow \exists J \in V,\ a \in J < b.$$

A Σ_1 formula $Y(x, y) = z$ is an indicator for X in M if

(i) $M \models \forall x, y \exists ! z \ Y(x, y) = z$, (we denote this function by Y^M or simply by Y when no confusion can arise),

(ii) for $a, b \in M$,

$Y^M(a, b) > N \Longleftrightarrow \exists J \in X, a \in J < b$

(iii) for $a_1, a_2, b_1, b_2 \in M$, if $a_1 \leqslant a_2$ and $b_2 \leqslant b_1$ then $Y^M(a_2, b_2) \leqslant Y^M(a_1, b_1)$.

We recall the following theorem concerning indicators.

<u>Theorem 0</u>. Let T be a rec. theory in the language of arithmetic and let $T \supseteq I\Sigma_1$. Suppose that $Y(x, y) = z$ is an indicator for $\{I \subseteq_e M | I \models T\}$ in any countable $M \models T$. Then

(i) $T \nvdash \forall x, z \exists y \ Y(x, y) \geqslant z$,

(ii) $\forall n \in N, T \vdash \forall x \exists y \ Y(x, y) \geqslant \underline{n}$,

(iii) if $N \models T$ then $\forall x, z \exists y \ Y(x, y) \geqslant z$ is independent of T,

(iv) if $f(x) = y$ is Σ_1 and $T \vdash \forall x \exists ! y \ f(x) = y$ then $\exists n \in N$ such that $T \vdash \forall x (f(x) < g_n(x))$, where $g_n(x) = \mu y : Y(x, y) \geqslant n$.

The proof of this result for $T = P$ is contained in [6], indeed the proof there shows that we can replace T by $T + R$ for any Π_1 set of sentences R consistent with T. The general result is proved similarly using for (ii) that if M is a countable recursively saturated model of $I\Sigma_1$ then M is isomorphic to a proper initial segment of itself. (See [1].)

Let $I \subseteq_e M$. We write $M \prec_I K$ if $M \prec_{\Sigma_0} K$, $I \subseteq_e K$ and for some $\pi \in K$, $I < \pi < (M-I)$. Say I is 1-extendible (in M) if $\exists K, M \prec_I K$ and I is $(n+1)$-extendible $(n > 0)$ if $\exists K, M \prec_I K$ and I is n-extendible in K. Write $I \models I\Sigma_n^*$ if for every Σ_n^0 formula $\theta(x, X_1, \ldots, X_m)$ in the second order language of arithmetic and $A_1, \ldots, A_m \subseteq I$ (coded by our convention)

$$I \models \exists x \theta(x, A_1, \ldots, A_m) \rightarrow \exists x(\theta(x, A_1, \ldots, A_m) \wedge \forall y < x \ \neg \theta(x, A_1, \ldots, A_m)).$$

Write $I \models B\Sigma_n^*$ if for all such Σ_n^0 formulae $\theta(x, y, X_1, \ldots, X_m)$ and $A_1, \ldots, A_m$ and $a \in I$,

$$I \models \forall x < a \exists y \theta(x, y, A_1, \ldots, A_m) \rightarrow \exists z \forall x < a \exists y < z \theta(x, y, A_1, \ldots, A_m).$$

$I \models I\Sigma_n$, $I \models B\Sigma_n$ denote the same properties for first order θ.

Let $[I]^k = \{<a_1, \ldots, a_k> | a_1, \ldots, a_k \in I \ \& \ a_1 < a_2 < \ldots < a_k\}$ (=I if $k = 1$). By using the standard pairing function we can code subsets of $[I]^k$ in M. $A \subseteq I$, not necessarily coded, is unbounded (in I) if $\forall a \in I \exists b > a, b \in A$. For $k \geqslant 1$, $A \subseteq [I]^{k+1}$, not necessarily coded, is unbounded (in $[I]^{k+1}$) if $\{a_0 | \{<a_1, \ldots, a_k> | <a_0, a_1, \ldots, a_k> \in A\}$ is unbounded$\}$ is unbounded. It is easy to see that if $1 \leqslant s < k+1$ then $A \subseteq [I]^{k+1}$ is unbounded if and only if $\{<a_0, \ldots, a_{s-1}> | \{<a_s, \ldots, a_k> | <a_0, \ldots, a_k> \in A\}$ is unbounded in $[I]^{k-s+1}\}$ is unbounded in $[I]^s$. I is k-Ramsey $(k \geqslant 1)$ if for all coded $f: [I]^k \rightarrow a$ with $a \in I$,