DATA REDUCTION (DARE) 
PRECONDITIONING FOR 
GENERALIZED CONJUGATE 
GRADIENT METHODS

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Abstract

The preconditioning of generalized conjugate gradient methods by means of data reduction (DARE) methods for linear systems arising from the finite difference method will be presented. DARE is a multilevel method where unknowns are dropped by using a functional approach. On each level iterations of generalized CG methods are performed according to different strategies. The better the functional approach fits the solution, i.e. the finer the size of the grid compared to the solution curvature, the better DARE works. Numerical experiments show that one gets a nearly linear increase in computation with the number of unknowns and that the convergence of DARE is rather independent from the operator of the partial differential equation. DARE turns out to be a promising and feasible technique, but a lot of work still has to be done looking for optimal strategies.

1 Introduction

We are concerned with the solution of 2-D and 3-D elliptic and parabolic systems of partial differential equations. They are solved numerically by the finite difference method of arbitrary consistency order by the black-box solver FIDISOL, see section 17 in [3]. The matrix resulting from the discretization and linearization of the system on a regular structured grid is large, sparse and it has the structure of scattered diagonals. Therefore the linear system is solved by generalized CG methods because they are optimally suited for the solution of this type of systems on vector computers [4]. As the solution of the linear system is the main expense of the whole problem solution and consumes about 90% of the computer time a good preconditioner is searched to accelerate the CG method. A preconditioner based on a multilevel method but different to multigrid methods [1] will be presented in the following. Our aim is to
solve the linear system

\[ Ax = b \]

where \( A \) is usually non-symmetric and not positive definite.

The matrix \( A \) results from a black-box solver that handles arbitrary differential operators with arbitrary boundary conditions. The consistency order of the finite difference method is selfadapted or chosen by the user. Therefore, nothing is known of the inner and the outer structure of the matrix. The task has been to develop a robust multilevel method that is applicable under such difficult conditions in contrast to a special multilevel method designed for a special differential equation with special boundary conditions and with a fixed consistency order.

The generalized CG methods have been chosen as solver because of their excellent vectorization properties in connection with the appropriate data structures. There is no closed theory for the different members of the CG family for non-symmetric, non-positive definite matrices. Consequently there is up to now no theory for DARE. But the promising numerical results of DARE shown in the following should encourage to more theoretical research.

2 The Generalized CG methods

Our investigations have been based on generalized CG methods.

Let be \( x_0 \) any initial guess for the solution of the system \( Ax=b \). The following recurrence is called pseudo-residual method (PRES).

Choose an "appropriate" \( d_k \)

\[ r_0 = Ax_0 - b , \]

\[ \bar{r}_{k+1} = \hat{A}d_k + \sum_{i=1}^{q} \alpha_{i,k}r_{k+1-i} , \]

where the \( \alpha_{i,k} \) are determined from \( \bar{r}_{k+1}^T \bar{r}_{k+1} = \min! \), i.e.

\[ \alpha_{i,k} = \frac{-r_{k+1-i}^T \hat{A}d_k}{r_{k+1-i}^T r_{k+1-i}} , \]

\[ r_{k+1} = \phi_k \bar{r}_{k+1} , \]

\[ x_{k+1} = \phi_k (r_k + \sum_{i=1}^{q} \alpha_{i,k} x_{k+1-i}) \]

with

\[ \phi_k = \frac{1}{\sum_{i=1}^{q} \alpha_{i,k}} . \]

\( q \) is called order of the PRES method.

The quantities \( \bar{r}_k \) are called pseudo-residuals. They have the same direction as the residuals but a different length. The minimum condition for the pseudo-residuals induces the orthogonalities \( \bar{r}_{k+1}^T \bar{r}_{k+1-i} = 0 \) for \( i=1,...,q \). The definition for the PRES