HIERARCHIES OF SETS AND DEGREES BELOW 0'

by

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We examine two hierarchies of sets below 0' based on the number of changes a recursive approximation to a set needs to make. Both are generalizations of the notion of being r.e. The first classifies sets by asking what functions dominate the number of changes, as previously set out in Epstein [4]. This extends the ideas of Putnam on trial and error predicates [8]. The second views the changes as dominated by a constructive ordinal, as first suggested by Addison [1], and developed by Ershov [6]. We provide a translation between them and relate these hierarchies to the degrees of unsolvability ≤ 0'.

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We first review some facts about sets ≤_T 0'. All notation comes from Epstein [4].

The reader should be aware that there is another hierarchy of degrees ≤ 0' which is based on the jump operator and is due primarily to Cooper. See Epstein [4], Chapter XI for that.

We indicate the end of a proof by □, and the end of a subproof by □.

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We notate $0' = \{(x,y): \varphi(y)\}$. Recall that

$0' \equiv_T K = \{x : \varphi_x(x)\}.$

First we note the Quantifier Characterization of Sets Below $0'$:

$A \subseteq_T 0'$ iff there are two recursive predicates $R,S$ such that $w \in A$ iff $\forall x \exists y R(x,y,w)$ iff $\exists x \forall y S(x,y,w)$

Proof: $\models \forall y S(x,y,w)$ is recursive in $0'$ since its the complement of an r.e. predicate. Therefore $\exists x \forall y S(x,y,w)$ is r.e. in $0'$. Similarly, $\exists x \forall y \neg R(x,y,w)$ is r.e. in $0'$. Thus $A$ and $\overline{A}$ are r.e. in $0'$ so $A$ is recursive in $0'$.

Let $f$ enumerate $0'$ and let $0'_s = \{x : f(y) = x < s, \text{ some } y < s\}$ be $0'$ enumerated to level $s$. For some $e$, $A = \phi_e(0')$ the $e^{th}$ function partial recursive in $0'$. Define $A_s(x) = \phi_{e,s}(0'_s)(x)$ (where $\phi_{e,s}$ is $\phi_e$ with calculations truncated by $s$). Then

$x \in A$ iff $\exists t s > t \phi_{e,t}(0')(x) = 1$

As a corollary to the proof we have

The Limit Lemma (Shoenfield [11]): $f \subseteq_T 0'$ iff there is some recursive function $g$ such that $f(x) = \lim_s g(s,x)$.

Here $\lim_s g(s,x) = f(x)$ means for all sufficiently large $s$, $g(s,x) = a = f(x)$. That is, $\exists t s > t g(s,x) = a$.

When we have $g(s,x)$ as in the Limit Lemma we call $g(s,x) = f_s(x)$ and call that a recursive approximation to $f$. If $f$ is 0-1 valued we'll view it as a set and say $g(s,x) = A_s(x)$.

If $A$ is r.e. then $A$ has an approximation $A_s$ such that $A_s(x)$ "changes its mind at most once." That is, there is at most one $s$ (if $x \in A$ there is one) such that $A_s(x) = 0$ and $A_{s+1}(x) = 1$ (if $x \notin A$, $A_s(x) = 0$ always).

The r.e. sets have always been considered a distinguished class of