Our main purpose in this paper is to establish model theoretic and invariant definability characterizations of recursion in $E$ and positive elementary induction (recursion in $E^\#$) on abstract structures. These theories are two different natural generalizations of the hyperarithmetic theory on the integers, while recursion in $E$ is in addition an extension of the theory of recursion in normal higher type objects.

Moschovakis [1969c, [EIAS] characterized the "boldface" hyperelementary (recursive in $E^\#$) relations using the Schema of $\Delta^1_1$-Comprehension. Here we study finer "lightface" analogs of this schema in terms of which we characterize both recursion in $E$ and positive elementary induction. Moreover we obtain hierarchies for these theories, which provide constructions of the recursive in $E$ and the recursive in $E^\#$ relations with the respective relations of lower levels as basis.

Due to the limitations of space, in this paper we restrict ourselves to the statements of the results and to a few comments about the tools used in the proofs. However in the first two sections we have included some of the basic definitions, as well as a survey of the results in the hyperarithmetic theory and the theory of positive elementary induction. The characterizations of recursion in $E$ are presented in §3, while the last section contains the results about recursion in $E^\#$ and a comparison between the two theories. The proofs of the main theorems and extensions of this work to recursion in generalized quantifiers will appear elsewhere.

§1. Recursion in $E$ and Recursion in $E^\#$ on a Structure

1.1. Let $\mathfrak{A} = \langle A, R_1, \ldots, R_n, f_1, \ldots, f_m, c_1, \ldots, c_q \rangle$ be a structure such that $\omega \subseteq A$ and for $k \in \omega$ let $\mathcal{P}_k^A$ be the set of all $k$-ary partial functions from $A$ into $\omega$. A functional (on $A$ with values in $\omega$) is a partial mapping

$$\Phi: A^k \times \mathcal{P}_{k_1}^A \times \ldots \times \mathcal{P}_{k_t}^A \rightarrow \omega$$

which is monotone, i.e., if $g_1 \subseteq h_1$, $g_2 \subseteq h_2$, ..., $g_t \subseteq h_t$ and $\Phi(x, g_1, \ldots, g_t) = \omega$, then $\Phi(x, h_1, \ldots, h_t) = \omega$.
If \( \Phi = (\Phi_1, \ldots, \Phi_s) \) is a sequence of functionals on \( A \), then we can define the notion of a \textit{recursive in} \( \Phi \) \( k \)-ary partial function from \( A \) into \( \omega \). Inductive definability provides a conceptually simple way to do this. We associate first with the sequence \( \Phi \) the class \( \mathcal{I} \Phi \), which is the smallest collection of functionals on \( A \) containing \( \Phi_1, \ldots, \Phi_s \) and satisfying certain minimal closure properties (closed under composition, definition by cases, functional substitution, etc.). Then the \textit{recursive in} \( \Phi \) partial functions are obtained by iterating to the transfinite the operative functionals in \( \mathcal{I} \Phi \) (i.e., the functionals in \( \mathcal{I} \Phi \) of the form \( \forall: A^s \times P_3^s \rightarrow \omega \)). The first few sections of Kechris-Moschovakis [1977] and Kolaitis [1978], [1979] contain a detailed account of these definitions.

A relation is \textit{semirecursive in} \( \Phi \) if it is the domain of a recursive in \( \Phi \) partial function; a relation is \textit{recursive in} \( \Phi \) if its characteristic function is recursive in \( \Phi \). The class \( \text{ENV}[\Phi] \) (the \textit{Envelope of} \( \Phi \)) is the collection of all semirecursive in \( \Phi \) relations, while the class \( \text{SEC}[\Phi] \) (the \textit{Section of} \( \Phi \)) is the collection of all recursive in \( \Phi \) relations. These notions relativize directly to a finite sequence \( \bar{x} = (x_1, \ldots, x_k) \) from \( A \), so that we put

\[
\text{ENV}[\Phi, \bar{x}] = \text{all semirecursive relations in } \Phi \text{ from } \bar{x},
\]
\[
\text{SEC}[\Phi, \bar{x}] = \text{all recursive relations in } \Phi \text{ from } \bar{x}.
\]

If we take the union of the above classes as \( \bar{x} \) varies over all finite sequences from \( A \), then we obtain respectively the collections of the "boldface" semirecursive in \( \Phi \) and "boldface" recursive in \( \Phi \) relations, namely

\[
\text{ENV}[\Phi] = \bigcup \text{ENV}[\Phi, \bar{x}]: \bar{x} \in A^{<\omega},
\]
\[
\text{SEC}[\Phi] = \bigcup \text{SEC}[\Phi, \bar{x}]: \bar{x} \in A^{<\omega},
\]

where \( A^{<\omega} \) is the set of all finite sequences from \( A \).

1.2. In this paper we are mainly concerned with the functionals \( E: P_3^1 \rightarrow \omega \) and \( E': P_3^1 \rightarrow \omega \) defined as follows:

\[
E(f) = \begin{cases} 
0, & \text{if } f \text{ is total and } (\exists x)(f(x) = 0), \\
1, & \text{if } f \text{ is total and } (\forall x)(f(x) \neq 0), \\
\text{undefined, if } f \text{ is not total};
\end{cases}
\]

\[
E'(f) = \begin{cases} 
0, & \text{if } (\exists x)(f(x) = 0), \\
1, & \text{if } (\forall x)(f(x) \neq 0), \\
\text{undefined, otherwise}.
\end{cases}
\]

Both these functionals embody existential quantification over \( A \). It is very easy to show that \( E \) is recursive in \( E' \), and therefore it is always true that