1. Introduction

The objective of this note is to begin the study of a fixed point index theory for fiber-preserving maps which will parallel the theory for self-maps of spaces, as set forth in [4] and [5], which emphasizes the role of the fundamental group. To review the global case, given a (smooth and compact, for simplicity) manifold $M$, one considers the inclusion map $i: M \times M \rightarrow M \times M$, where $\Delta M$ is the diagonal in $M \times M$, and replaces it by a fiber map $q: E \rightarrow M \times M$. The fiber $F = q^{-1}(\ast)$ in this case in $(m - 2)$-connected where $m = \dim M$ and, in particular, when $m \geq 3$, $F$ is simply connected. Furthermore, one easily establishes isomorphisms

$$\pi_{m-1}(F) \cong \pi_m(M \times M, M \times M - \Delta M) \cong \pi_m(M, M - y_0) \cong \mathbb{Z}[\pi],$$

where $y_0 \in M$ and $\pi = \pi_1(M)$. If we designate by $B$ the local system on $M \times M$ with local group $\pi_{m-1}(F) \cong \mathbb{Z}[\pi]$, the (right) action of $\pi \times \pi$ on $\mathbb{Z}[\pi]$ which determines $B$ is given by the formula

$$a \circ (\sigma, \tau) = (\text{sgn } \sigma) \sigma^{-1} a \tau,$$

where $\text{sgn } \sigma = \pm 1$ according as $\sigma \in \pi$ preserves or reverses a local orientation of $M$. Now, given a self map $f: M \rightarrow M$ one considers the diagram

$$
\begin{align*}
E(f) \quad & \xrightarrow{q_f} \quad E \\
M \quad & \xrightarrow{1 \times f} \quad M \times M
\end{align*}
$$
where \((E(f), q_f, M)\) is the fiber space induced from \((E, q, M \times M)\) by the map \(1 \times f\). A geometric argument shows that \(q_f\) admits a cross section if, and only if, \(f\) is deformable to a fixed point free map \(g : M \to M\). If we let \(B(f)\) denote the local system on \(M\) induced from \(B\) by \(1 \times f\), then the local group of \(B(f)\) is \(\mathbb{Z}[\pi]\) with \(\pi\) acting on \(\mathbb{Z}[\pi]\) by

\[
\alpha \circ \sigma = (\text{sgn} \, \sigma) \, \sigma^{-1} \alpha \varphi(\sigma), \quad \alpha, \sigma \in \pi
\]

where \(\varphi : \pi \to \pi\) is induced by \(f\). Let \(o(f) \in H^m(M; B(f))\) denote the primary obstruction to finding a cross section to \(q_f\). \(o(f)\) is invariant under homotopy and consequently a necessary and sufficient condition that \(f\) is deformable to a fixed point free map \(g : M \to M\) is that \(o(f) = 0\). Thus, \(o(f)\) serves as a (global) fixed point index for \(f\). (There is, of course, the corresponding local version [5]). \(o(f)\) is computable in terms of Nielsen classes as follows: First, one introduces the action \(\alpha \ast \sigma = \varphi(\sigma^{-1}) \alpha \sigma\) of \(\pi\) on \(\mathbb{Z}[\pi]\) and then let \(R(f)\) denote the corresponding local system. If \(T(M)\) is the orientation sheaf on \(M\), there is a dual pairing

\[B(f) \otimes T(M) \to R(f)\]

given by \(\alpha \otimes 1 \leftrightarrow \alpha^{-1}\) which induces a cap product

\[\langle , \rangle : H^m(M; B(f)) \otimes H_m(M; T(M)) \to H_o(M, R(f))\]

where \(H_o(M, R(f)) \approx \mathbb{Z}[\varphi]\), and \(\mathbb{Z}[\varphi]\) is the space of orbits (usually called Reidemeister classes) of \(\pi\) under the action \(\ast\).

**Theorem**

If \(\mu\) is the twisted integral fundamental class of \(M\), then

\[\langle o(f), \mu \rangle = \sum_{\rho \in R} \lambda_{\rho} \rho\]

where \(R = \mathbb{Z}[\varphi]\), and \(\lambda_{\rho}\) is the local numerical index of the Nielsen class corresponding to the Reidemeister class \(\rho\). In particular, the Lefschetz number \(L(f)\) of \(f\) is the sum of the coefficients \(\lambda_{\rho}\).

We will consider in this note the following setting. Let \(\xymatrix{Y \ar[r]^-i & M \ar[r]^-p & B}\) denote a smooth fiber bundle and let