ON A CONJECTURE OF HOPF FOR \( \alpha \)-SEPARATING MAPS
FROM MANIFOLDS INTO SPHERES

By

FRIEDRICH VILLE
Fachbereich Mathematik
Universität Kassel
Heinrich-Plött Strasse 41
3500 Kassel, West Germany.

1. INTRODUCTION

Let \( f : M \to X \) be a continuous map from a metric space \( M \) into a topological space \( X \). Assume that there exists a real number \( \alpha > 0 \) satisfying

\[
(x_1, x_2 \in M \text{ and } d(x_1, x_2) = \alpha) \implies f(x_1) \neq f(x_2).
\]  \hspace{1cm} (1)

(\( d \) denotes the metric of \( M \)). A map with this property will be called a \( \alpha \)-separating map.

In this paper we study the following case: assume \( X = S^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \} \) being the \( n \)-dimensional sphere and \( M = M^n \) a smooth compact connected oriented \( n \)-dimensional manifold with a Riemannian metric \( d \). Furthermore let \( \alpha \) be a positive real number such that for any two points \( x_1, x_2 \in M^n \) with \( d(x_1, x_2) = \alpha \) there is a unique minimal geodesic from \( x_1 \) to \( x_2 \). Considering this \( \alpha \) let

\[
f : M^n \to S^n
\]

be a \( \alpha \)-separating map. We will prove the following.

THEOREM

Assuming \( f \) as above, the topological degree of \( f \) does not vanish:

\[
\deg f \neq 0.
\]

This theorem was conjectured by H. Hopf [4, p. 136-137] in 1945. Especially
he noted that even in the case \( M^n = S^n \) (\( n \geq 2 \)) the result is still unknown. He remarked that the theorem is obviously true for \( n = 1, M^1 = S^1 \). In [2, 3] G. Hirsch proved \( \deg f \neq 0 \) under the strong additional assumption that

\[
f(x_1) \neq f(x_2)
\]

if \( d(x_1, x_2) = a \). In [6, 7, 8] the writer proved the theorem for \( M^n = S^n \), \( n \) even, and in the case \( a = n/2 \) for all \( n \). In a recent paper [1] T. tom Dieck and L. Smith gave the following result: under the assumptions above the Euler characteristic of \( M^n \) is even and the following congruences are true: \( \deg f = \chi(M^n)/2, \mod 2 \), if \( n \) is even, and \( \deg f = \chi_{1/2}(M^n), \mod 2 \), if \( n \) is odd and \( n \neq 1, 3, 7 \).

(\( \chi \) denotes the Euler characteristic and \( \chi_{1/2} \) the Kervaire semi-characteristic). This gives an affirmative answer to the conjecture in the case \( M^n = S^n \) for all \( n \) except \( n = 3 \) and \( n = 7 \). In the following sections the theorem will be proved solving the problem of Hopf.

2. Basic Conditions

Let us fix \( \alpha \) and \( f \) as above and assume \( n \geq 2 \). Defining the ball

\[
K_{p, \alpha} := \{ x \in M^n : d(x, p) \leq \alpha \}, \quad p \in M^n,
\]

one obtains

**Lemma 1**

The topological degree

\[
\delta := \deg(f, K_{p, \alpha}, f(p))
\]

is odd and independent from \( p \in M^n \).

**Proof**

The oddness of \( \delta \) follows from [8] (proof of Lemma 3). The independence of \( p \), we get from the homotopy invariance of the topological degree in the fol-