CHAPTER II

RELATIONS BETWEEN EQUIVARIANT SURGERY THEORIES

... laßt uns angenehmere anstimmen ...
(Recitative, Beethoven's Ninth Symphony)

Although it is clear that the various approaches to equivariant surgery are based upon related ideas, there is relatively little in the literature to describe the means for passing from one theory to another. At several points in this book we shall see that the conclusions from different versions of equivariant surgery theory shed a great deal of light on each other, and consequently an explicit description of the relationships is useful for technical as well as expository purposes.

It is often extremely helpful to view the orbit space of a manifold with group action as a manifold with singularities. In particular, the Browder-Quinn approach to equivariant surgery in [BQ] is based explicitly upon a canonical description of a smooth action's orbit space as a stratified set in the sense of Thom and Mather. Although the existence of this stratification is mentioned and applied repeatedly in the literature, an easily accessible account of the proof does not exist. The standard reference has been W. Lellmann's unpublished Diplomarbeit [Le], and the most descriptive comments in a published work appear to be on page 21 of [G+]. For the sake of completeness we shall explain in Section 4 how a proof of the stratification theorem can be extracted from the standard papers and books on stratified sets.

Throughout this chapter $G$ will denote a compact Lie group unless indicated to the contrary.

1. Browder-Quinn Theories

We shall begin by recalling two of the main points from the first section of Chapter I:

(i) A free action of $G$ on a space $X$ is completely determined by the orbit space $X/G$ and some additional data (i.e., the homotopy class of the classifying map $X/G \to BG$).

(ii) Frequently one can obtain useful information about smooth, nonfree $G$-actions by applying surgery theory to the nonsingular part of a smooth orbit space $M^* = M/G$. 
There are many other situations in transformation groups where group actions are completely determined by orbit spaces together with some additional structural data. During the nineteen sixties many special cases were studied extensively. A survey article by G. Bredon from that period [Bre] describes several of these examples and their applications to exotic group actions on spheres. Subsequent work has shown that smooth actions of compact connected Lie groups on 3- and 4-manifolds are completely determined by so-called \textit{structured} or \textit{weighted} orbit space data (e.g., see [Ray] for circle actions on 3-manifolds, and see [OR1-2], [Fi1-2], and [Prkr] for group actions on 4-manifolds); analogous results also exist for certain group actions on higher dimensional manifolds. Such results suggest that a suitable extension of surgery theory to orbit spaces of group actions might yield new insights and further information about nonfree actions. In [BQ] Browder and Quinn showed that such a theory could be constructed as a special case of a \textit{surgery theory for stratified sets}. The purpose of this section is to describe the main features of their theory, paying particular attention to the properties that will be needed later in this book.

\textit{Browder-Quinn stratification data}

The \textit{strata} of a smooth $G$-manifold were discussed briefly in Section 2 of Chapter I; some of the definitions will be repeated here for the sake of convenience and clarity. If $M$ is a smooth $G$-manifold and $H$ is a closed subgroup of $G$, then the local linearity of smooth actions implies that

$$M(H) = \{ x \in M \mid \text{the isotropy subgroup } G_x \text{ is conjugate to } H \}$$

is a (possibly empty!) disjoint union of smooth $G$-invariant submanifolds of

$$M = \bigcup_{K \supseteq H} M(K)$$

(where different components may have different dimensions), and the projection

$$\pi_H : M(H) \rightarrow M(H)/G = M^*_H$$

is a fiber bundle with fiber $G/H$ and structure group $N(H)/H$; in fact, this is a smooth fiber bundle over each component of $M^*_H$. For each $x \in M(H)$ one has a tubular neighborhood in $M$ of the form $G \times H V(x)$ for some $H$-representation $V(x)$. If $W$ is an arbitrary linear representation of $H$, then

$$M_{H,W} = \{ x \in M(H) \mid V(x) \approx_H W \}, \quad M^*_H,W = M_{H,W}/G$$

define unions of components of $M(H)$ and $M^*_H$ respectively. The advantages of these smaller subsets are that each such set is a smooth manifold and the restrictions of the orbit space projection defines a smooth bundle $M_{H,W} \rightarrow M^*_H,W$ for every $H$ and $W$. The sets $M^*_H,W$ are called the \textit{strata} of $M/G$ associated to the original group action; notice that every point of $M/G$ belongs to exactly one stratum $M^*_H,W$. Frequently