CHAPTER V

PRODUCTS AND PERIODICITY FOR SURGERY UP TO PSEUDEQUIVALENCE

Most of the equivariant surgery theories in Chapters I-IV are designed to study group actions that are more or less similar to some given example. However, a few equivariant surgery theories are designed to yield very exotic group actions with few similarities to obvious models. The main examples of such theories are the theories $I^h$ and $I^h(R)$ for equivariant surgery up to pseudoequivalence that are mentioned in (11.2.5) and developed in [DP1-2] and [PR]; the power of such theories is illustrated by their applications to group actions on spheres with one fixed point [Pe2] and numerous other questions (see [DPS] for further information). In order to study existence questions for very exotic actions it is necessary to study extremely general classes of equivariant surgery problems, and the required level of generality leads to many new technical complications. It is more or less predictable that some of these will cause difficulties in any attempt to extend the periodicity theorems of Chapter III to the theories $I^h$ and $I^h(R)$. In this chapter we shall study the difficulties that arise and deal with them effectively enough to obtain periodicity theorems for the theories $I^h(G; -)$ and $I^h(R)(G; -)$ in many important cases; for example, there are periodicity theorems if $G$ is an odd order abelian group. We shall also consider some variants of the basic theories $I^h$ and $I^h(R)$ from [DP1-2].

Here is a more specific description of this chapter’s contents. In Section 1 we shall describe the indexing data and normal maps for $I^h$ and $I^h(R)$, and in Section 2 we shall state the main results on the existence, additivity, and periodicity properties of product constructions for these theories. The next four sections (3 through 6) deal with formal properties of the $I^h$ and $I^h(R)$ obstruction groups such as the stepwise obstruction homomorphisms, the maps given by restriction to closed substrata, and the properties of certain projective class group obstructions that do not appear in the other equivariant surgery theories considered in this book. Products in $I^h$ and $I^h(R)$ are discussed in Section 7, and in Section 8 we combine the results of Sections 3–7 with the methods of Chapter III to prove the main results from Section 2. Finally, in Section 9 we prove a vanishing theorem for Wall groups that underlies the entire approach of this chapter; namely, if $G$ is an odd order group and $R$ is a subring of the rationals, then the odd (homotopy) Wall groups of the group ring $R[G]$ are trivial. It seems likely that this result had been known to some researchers in the area, but the literature does not seem to contain a result stated at the required level of generality. A reader who is primarily interested in the main results should begin by reading Sections 1, 2, and 8; the remaining material is more of a technical nature.
1. The setting

As indicated in the introduction, this chapter deals mainly with the theories \( I^h \) and \( I^{h(R)} \) for surgery up to pseudoequivalence described in [DP1–2] and [PR]. These theories are very similar to the theory \( I^{h,DIF} \) of Chapter I and [DR1] in many respects, but there are numerous complications. As noted in (II.2.5) the first complications arise in formulating an appropriate notion of indexing data, and there are further difficulties in defining the sorts of groups, manifolds, and mappings to be considered. In this section we shall deal with the main technical complications as part of our summary of the theories \( I^h \) and \( I^{h(R)} \) for actions of odd order groups. At the end of this section we shall describe the modifications needed to treat even order groups. Fortunately, a relatively limited discussion of technicalities will suffice to formulate the most basic definitions and prove what we need, and it will not be necessary to mention many subtle and unpleasant points. The complete references for this material are [DP1, Sections 1–4, 9, and Appendix A–1] and [PR].

Indexing data

We shall assume throughout that \( G \) is a finite group and \( f : M \to X \) is an equivariant map of compact, smooth \( G \)-manifolds. Furthermore, we shall assume that all smooth \( G \)-manifolds satisfy the Codimension \( \geq 2 \) Gap Hypothesis of Section I.2 unless indicated otherwise.

In Section I.2 we defined the indexing data

\[
\lambda_M = (\pi(M), d_M, s_M, w_M) \\
\lambda_X = (\pi(X), d_X, s_X, w_X)
\]

associated to \( M \) and \( X \). Specifically, the geometric posets \( \pi(M) \) and \( \pi(X) \) list the components of the various fixed point sets and their interrelations, the dimensions \( d_M \) and \( d_X \) give the dimensions of these components, the normal slice data \( s_M \) and \( s_X \) describe the oriented normal slices on the sets in \( \pi(M) \) and \( \pi(X) \), and the orientation data \( w_M \) and \( w_X \) describe the group action’s effect on orientations over each set in \( \pi(M) \) and \( \pi(X) \). For the sake of relative simplicity we shall assume that \( M \) and \( X \) satisfy the Codimension \( \geq 3 \) Gap Hypothesis and all closed substrata of \( M \) and \( X \) (in the sense of Section I.2) are oriented. The functional indexing data of \( f \) will be the \( r \)-tuple

\[
\lambda(f) = (\lambda_M, \lambda_X, \hat{f}, \mu)
\]

where \( \lambda_M \) and \( \lambda_X \) are as before, the map \( \hat{f} : \pi(M) \to \pi(X) \) is the induced map of \( G \)-posets described in Section I.2, and \( \mu : \pi(X) \to \mathbb{Z} \) is degree data given by

\[
\mu(\beta) = \sum \text{degree } (f_\alpha : M_\alpha \to X_\beta),
\]

the sum running through all \( \alpha \) such that \( \hat{f}\alpha = \beta \) and \( \dim M_\alpha = \dim X_\beta \) (compare [DP1, (1.12), p. 5]).