The Artin-Rees Property

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All rings are assumed to be associative but not necessarily commutative and to have an identity element 1. All modules are right modules and are unital. In this paper we review some recent work concerning the weak Artin Rees property, called the AR property for short.

1. The Artin-Rees Theorem

1.1 Theorem (Artin, Rees). Let \( R \) be a commutative Noetherian ring, \( I \) an ideal of \( R \), \( M \) a finitely generated \( R \)-module and \( N \) a submodule of \( M \). Then there exists a positive integer \( k \) such that for all \( n \geq k \)

\[
N \cap M^i = (N \cap M^k)I^{n-k}.
\]

Proof (See (29)). Suppose first that \( M = R \). There exists a finite collection of elements \( a_1, \ldots, a_s \) in \( I \) such that \( I = Ra_1 + \ldots + Ra_s \). Let \( x \) be an indeterminate and \( S \) the subring of the polynomial ring \( R[x] \) generated by \( R \) and the elements \( a_i x (1 \leq i \leq s) \). Then \( S \) consists of all polynomials of the form

\[
c_0 + c_1 x + \ldots + c_n x^n
\]

where \( n \geq 0 \), \( c_0 \in R \) and \( c_j \in I^j (1 \leq j \leq n) \). If \( N \) is an ideal of \( R \) let \( E \) be the ideal

\[
N + (N \cap I)x + (N \cap I^2)x^2 + \ldots
\]
of \( S \), i.e. \( E \) consists of all polynomials

\[
c_0 + c_1 x + \ldots + c_n x^n
\]

such that \( n \geq 0 \), \( c_0 \in \mathbb{N} \) and \( c_j \in \mathbb{N} \cap I^j \) \((1 \leq j \leq n)\). By Hilbert's Basis Theorem the commutative ring \( S = R[\alpha_1 x, \ldots, \alpha_s x] \) is Noetherian. Thus the ideal \( E \) is finitely generated, say by elements \( f_1(x), \ldots, f_m(x) \).

Define \( k = \max_i \deg f_i(x) \). Let \( n \geq k \). Let \( a \in \mathbb{N} \cap I^n \). Then \( ax^n \in E \) and so

\[
ax^n = b_1(x)f_1(x) + \ldots + b_m(x)f_m(x)
\]

for some \( b_1(x), \ldots, b_m(x) \) in \( S \). Comparing coefficients of \( x^n \) we see that

\[
a \in \sum_{t=0}^{k} I^{n-t} (\mathbb{N} \cap I^t)
\]

where we make the convention \( I^0 = R \). But, for \( 0 \leq t \leq k \),

\[
I^{n-t} (\mathbb{N} \cap I^t) = I^{n-k} I^{k-t} (\mathbb{N} \cap I^t)
\]

\[
\leq I^{n-k} (\mathbb{N} \cap I^k).
\]

Thus \( a \in (\mathbb{N} \cap I^k)I^{n-k} \). It follows that \( \mathbb{N} \cap I^n \leq (\mathbb{N} \cap I^k)I^{n-k} \). The converse is clear and so \( \mathbb{N} \cap I^n = (\mathbb{N} \cap I^k)I^{n-k} \), and this is true for all \( n \geq k \).

In general, let \( M \) be a finitely generated \( R \)-module. Define \( T \) to be the set of \( 2 \times 2 \) "matrices"

\[
\begin{bmatrix}
 r & m \\
 0 & r
\end{bmatrix}
\]

with \( r \) in \( R \) and \( m \) in \( M \). Matrix addition and multiplication make \( T \) a ring. Moreover, \( T \) is a commutative Noetherian ring. Define

\[
J = \left\{ \begin{bmatrix}
 a & m \\
 0 & a
\end{bmatrix} : a \in I, m \in M \right\}
\]