1. Introduction. By a $\delta$-fraction we mean a finite or infinite continued fraction of the form

$$b_0 - \delta_0 z + \frac{d_1 z}{1-\delta_1 z} + \frac{d_2 z}{1-\delta_2 z} + \frac{d_3 z}{1-\delta_3 z} + \cdots,$$

where $b_0$ and $d_n$ are complex constants, $d_n \neq 0$ for $n \geq 1$, and the $\delta_n$ are real constants restricted to the values 0 or 1. We adopt the convention that the $\delta$-fraction (1.1), and all of its approximants, have value $b_0$ at $z = 0$. We say that the $\delta$-fraction (1.1) is regular if $d_{k+1} = 1$ for each $k$ such that $\delta_k = 1$. We choose the name $\delta$-fraction for the continued fraction (1.1) because of the binary "impulse" nature of the sequence $\{\delta_n\}$ and the analogies, therefore, with the $\delta$'s in the Dirac delta function and the Kronecker delta symbol. We are led to this investigation of $\delta$-fractions, and their connections with analytic functions, in our quest to find an answer to the following question:

Is there a class $D$ of "simple" continued fractions

$$b_0(z) + \frac{a_1(z)}{b_1(z)} + \frac{a_2(z)}{b_2(z)} + \frac{a_3(z)}{b_3(z)} + \cdots,$$

having the following desirable properties?

(a) the elements $a_n(z)$ and $b_n(z)$ are polynomials in $z$ of degree $\leq 1$.

(b) $D$ contains the class of regular C-fractions

$$c_0 + \frac{c_1 z}{1} + \frac{c_2 z}{1} + \frac{c_3 z}{1} + \cdots, \quad c_n \in \mathbb{C}, \quad c_n \neq 0 \text{ if } n \geq 1.$$

(c) Given the power series

$$L_0 \equiv c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad c_n \in \mathbb{C}, \quad c_0 \neq 0,$$

there exists a unique member $K_0 \in D$ such that $K_0$ corresponds to $L_0$, i.e., the Maclaurin series of the $n$-th approximant of $K_0$ agrees termwise with $L_0$ up to and including the term $c_{k(n)} z^{k(n)}$, where $k(n) \to \infty$ as $n \to \infty$.

(d) If $L_0$ represents a rational function in a neighborhood of $z = 0$, then its corresponding $K_0 \in D$ terminates.

(e) Let us say that $K \in D$ corresponds to a function $f(z)$, analytic at $z = 0$, if $K$ corresponds to the Maclaurin series of $f$. Then for many classical functions, analytic in a neighborhood of the origin, explicit and useful formulas can be obtained for the elements $a_n(z)$ and $b_n(z)$ of the continued fractions in $D$ corresponding to these functions.

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(f) Much information can be given about the convergence of the continued fractions in $D$ that correspond to functions which are analytic at the origin.

(g) In many cases, at least "some" of the approximants of the continued fraction $K \in D$ corresponding to a power series $L_0$ are in the Padé table for $L_0$.

The C-fractions of Leighton and Scott [13], the T-fractions of Thron [26], and the P-fractions of Magnus [14,15] all essentially meet requirement (c), but each of these classes fails to meet one or more of the remaining requirements. The C-fractions in general do not meet requirement (a), and the regular C-fractions above do not meet requirement (c). For example, it is known that there is no regular C-fraction corresponding to $1 + z^2$. The T-fractions essentially meet requirements (a) and (f), in addition to (c), but they do not meet (b), (d), and (g). The P-fractions have a close connection with the Padé table of a given power series $L_0$, but they fail to meet requirements (a) and (b), among others. The class of general T-fractions, studied by Waadeland [31,32] and others, essentially contains our class of $\delta$-fractions, but, to date, this general class has been studied more from the interpolation point of view, i.e., more from the point of view of their connections with two-point Padé tables for meromorphic functions. Closely related to the general T-fractions are the M-fractions studied by a number of authors [2], [17], [18]. For thorough treatments of the subject of correspondence and the properties of many kinds of corresponding continued fractions, including the ones just mentioned, we refer the reader to the recent book on continued fractions by Jones and Thron [9].

We offer the class of regular $\delta$-fractions as our candidate for an answer to the question posed above. It is easily seen that these continued fractions satisfy (a) and (b). In Section 2 we give five basic theorems dealing with the correspondence between $\delta$-fractions and power series. It follows from Theorems 2.1 and 2.4, respectively, that requirements (c) and (d) are satisfied if $D$ denotes the class of regular $\delta$-fractions. The proof of Theorem 2.4 turns out to be considerably more complicated than the proof given by Perron [21, page 111] of the corresponding result for C-fractions.

In Section 3 we make a start towards meeting requirement (f) with our $\delta$-fractions, by offering four convergence theorems involving these continued fractions. At the beginning of the section we state, in Theorem A, a version of Poincaré's Theorem on linear homogeneous difference equations, which we use later. Theorem 3.1 is concerned with properties of uniformly convergent $\delta$-fractions. Theorem 3.2 essentially states that the $\delta$-fraction (1.1) converges to an analytic function in a neighborhood of $z = 0$, if the sequence $(d_n)$ is bounded. It follows from Theorem 3.3 that the $\delta$-fraction (1.1) converges to a meromorphic function in the unit disk $|z| < 1$, if $\lim_{n \to \infty} d_n = 0$. In Section 3 we also introduce the concept of a $(p,q)$ limit periodic $\delta$-fraction. Theorem 3.4 is a very useful convergence theorem for limit periodic $\delta$-fractions.