Chapter V

Invariant Theory, Arithmetic and Vector Bundles

§22 The Mysterious Role of Invariant Theory

In §20 we saw how the group of symmetries of the modular curve helped to make the study of the equations 19.13 more tractable from our elementary point of view. In §21, an invariant of degree 4 for the symplectic group dominated the study of the moduli space of theta structures of level $\delta = (3,3,\ldots,3)$ on abelian varieties. In this section, we discuss more examples of this phenomenon and try to formulate a general perspective.

Example (22.1): At the end of §20, we gave the equation of the modular curve of level 7 as

$$X^3Y + Y^3Z + Z^3X = 0$$

in $\mathbb{P}^2$. The left hand side is the unique quartic invariant$^9$ for $PSL_2(\mathbb{F}_7)$ in this representation.

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$^9$ This quartic was discovered by Felix Klein. Klein went further and determined the full ring of invariants for a 3 dimensional irreducible complex representation of $PSL_2(\mathbb{F}_7)$. For details see [K1] and [K-F]. One can give simple proofs that this quartic is the unique quartic invariant for $PSL_2(\mathbb{F}_7)$. First, a character computation shows that there exists a unique quartic invariant for this group. Either from the character or from the explicit description of the representation as the odd part of the Weil representation, one knows that with respect to a suitable basis, the upper triangular subgroup of order 21 acts in the following way: it maps to the group generated by the diagonal matrix with entries $[\zeta, \zeta^4, \zeta^2]$, where $\zeta$ is a primitive 7th root of unity, and by the cyclic permutation of order 3 of the coordinates. The element of order 7 has for its invariant quartics the linear combinations of the 3 monomials $X^3Y$, $Y^3Z$ and $Z^3X$. It is then obvious that Klein’s quartic is, up to a constant multiple, the only linear combination of these 3 which is invariant under the cyclic permutation. A similar argument shows that (22.4) is the unique cubic invariant for $PSL_2(\mathbb{F}_{11})$ acting on $\mathbb{C}^5$. 
Example (22.3): The following matrix

\[
\begin{pmatrix}
w & v & 0 & 0 & z \\
v & x & w & 0 & 0 \\
0 & w & y & x & 0 \\
0 & 0 & x & z & y \\
z & 0 & 0 & y & v
\end{pmatrix}
\]

is, up to a factor of 2, the matrix of second partial derivatives of a cubic invariant for $PSL_2(\mathbb{F}_{11})$, namely

\[(22.4) \quad v^2w + w^2x + x^2y + y^2z + z^2v = 0.\]

The modular curve is then the singular locus of the Hessian of this cubic threefold.

This result was discovered by Felix Klein [K2], [K-F]. Closer examination of his work shows that he actually did not distinguish carefully between the rank 3 locus of the matrix of second partials and the singular locus of the Hessian as a whole. That they are in fact the same is shown in [A2], [A16].

Example (22.5): A Kummer surface is the image in $\mathbb{P}^3$ of an abelian surface under the linear system $\Gamma(L)$ where $L$ is an invertible sheaf of level $\delta = (2,2)$. The Heisenberg group $G(L)$ acts on $\mathbb{P}^3$ and the Kummer surface is a quartic invariant.

Example (22.6): Let $E$ be an elliptic normal curve in $\mathbb{P}^4$. Then $E$ has degree 5 and is invariant under the Heisenberg group $G(5)$. We know from the Riemann-Roch theorem that $E$ lies on 5 quadrics and in fact these quadrics can easily be shown to define $E$. Let $\phi_0, \phi_1, \phi_2, \phi_3, \phi_4$ be a basis for the quadrics containing $E$. If $x = [x_0, x_1, x_2, x_3, x_4]$ is a point of $\mathbb{P}^4$, denote by $Q_x$ the quadric

\[(22.7) \quad \sum_{i=0}^{4} x_i \phi_i = 0\]

in $\mathbb{P}^4$. Let $J$ denote the quintic hypersurface in $\mathbb{P}^4$ consisting of all $x$ such that $Q_x$ is singular. Then $J$ is an invariant for the action of the Heisenberg group. In fact, $J$ is the chord locus of the elliptic normal curve $E$ and the singular locus of $J$ is $E$. The first author believes he found this example in a work of Halphen, but for some reason has been unable to find it again.

Example (22.8): In their joint paper [N-R 1], Narasimhan and Ramanan have the following example. Let $J$ denote the Jacobian variety of a curve of genus 3 which is not hyperelliptic. We can map $J$ into $\mathbb{P}^7$ using twice the theta divisor. The image is the Kummer variety considered by J. Coble [Co2], who suggested that the Kummer variety might be the singular locus of a certain quartic invariant for the Heisenberg group $G(2,2,2)$. Narasimhan and Ramanan were able to prove this using their results on moduli of rank 2 vector bundles.

Example (22.9): Let $X$ be the Jacobian variety of a curve $C$ of genus 2 and let $L = \mathcal{O}_X(C)$. Consider the of $X$ into $\mathbb{P}^8$ determined by $L^3$. By a theorem of Lefschetz, we know that this mapping is an embedding. We will therefore identify $X$ with its image. The locus $X$ is invariant under the Heisenberg group $G(3,3)$ and