3 Representations: homology and homotopy

We note here representations of $\mathcal{E}(X)$ and $\mathcal{E}^*(X)$ into automorphisms of homotopy and local-coefficient homology and related functors, and introduce some notation which will be used later. The special case of nilpotent spaces is noted: in this case we have representations into products of general linear groups $GL(\mathbb{Z}, n)$ provided that the homotopy groups have finite rank in each dimension.

Homotopy and integral homology representations give the following exact sequences

\begin{align*}
0 & \to K^*_\pi \to \mathcal{E}^*(X) \to \text{aut } \pi_1(X), \quad \text{with image } I^*_\pi(X), \\
0 & \to K^*_H \to \mathcal{E}^*(X) \to \text{aut } H_1(X), \quad \text{with image } I^*_H(X), \\
0 & \to K^*_\pi \to \mathcal{E}^*(X) \to \prod \text{aut } \pi_r(X), \quad \text{with image } I^*_\pi(X), \\
0 & \to K^*_H \to \mathcal{E}^*(X) \to \prod \text{aut } H_r(X), \quad \text{with image } I^*_H(X),
\end{align*}

The representation $\mathcal{E}^*(X) \to \text{aut } \pi_1(X)$ is onto if $X$ is a finite 2-dimensional CW complex with only one 2-cell or more generally the cellular model of an irreducible combinatorial aspherical presentation (see §15.2.3), but is not onto in general (see for example §15.2.1 and §15.3.1). For integral homology the representations factor through $\mathcal{E}(X)$. A corresponding factorisation for homotopy is given below. The elements of $I^*_H(X)$ are called the \textit{geometrically realizable} automorphisms of $H^*(X)$, and similar terminology applies in the other cases.

Given that the integer-homology is of finite rank $n_r$ in each dimension $r$, then, as in Proposition A.4 in the Appendix, there is the surjective representation $\prod \text{aut } H_r(X) \to \prod GL(n_r, \mathbb{Z})$ onto a product of general linear groups; and, moreover, the kernel of this representation is finite in the case where the torsion homology is finitely generated. In the case where the fundamental group is abelian, similar observations apply to the representation of $\prod \text{aut } \pi_r(X)$ onto a product of general linear groups.

We now note finer homotopy representations for $\mathcal{E}^*(X)$ and $\mathcal{E}(X)$ in the non-simply-connected case: these are used extensively in calculations (see Chapter 13). We consider the homotopy groups $\pi_r(X)$ ($r \geq 2$) as left $\mathbb{Z}\pi_1$-modules where $\pi_1 = \pi_1(X)$. We define $\text{aut } \pi_1(X) \times_{\phi} \text{aut } \pi_r(X)$ to be the subgroup of $\text{aut } \pi_1(X) \times \text{aut } \pi_r(X)$ consisting of pairs $(\alpha, \beta)$ for which the following diagram is commutative.
Let $D_r(\pi_1(X))$ be the image of the diagonal $\pi_1(X) \to \text{aut} \pi_1(X) \times_\phi \text{aut} \pi_r(X)$ under the (left) action of the fundamental group $\pi_1(X) \to \text{innaut} \pi_1(X) \to \text{aut} \pi_1(X)$ given by $g \sim (\cdot)^g$, where $(h)^g = g + h - g$, and $\phi : \pi_1(X) \to \text{aut} \pi_r(X)$ (see Appendix B). In the case where $X$ satisfies the conditions of Proposition 1.1, we now have the following diagram of homomorphisms.

\[
\begin{array}{ccc}
\pi_1(X) & \longrightarrow & \mathcal{E}^*(X) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & D_r(\pi_1(X)) \\
\end{array}
\]

We thus have the representations

\begin{align}
\mathcal{E}^*(X) & \to \text{aut} \pi_1(X) \times_\phi \text{aut} \pi_r(X) \quad (r \geq 2), \\
\mathcal{E}^*(X) & \to \text{aut} \pi_1(X) \times_\phi \prod_{r \geq 2} \text{aut} \pi_r(X), \\
\mathcal{E}(X) & \to \text{out} \pi_1(X) = \text{aut} \pi_1(X)/\text{innaut} \pi_1(X), \\
\mathcal{E}(X) & \to (\text{aut} \pi_1(X) \times_\phi \text{aut} \pi_r(X))/D_r(\pi_1(X)) \quad (r \geq 2), \\
\mathcal{E}(X) & \to \left(\text{aut} \pi_1(X) \times_\phi \prod_{r \geq 2} \text{aut} \pi_r(X)\right)/D(\pi_1(X)), \\
\mathcal{K}_{\pi_1}(X) & \to \left(\prod_{r \geq 2} \text{aut} \pi_1 \pi_r(X)\right)/\phi(3(\pi_1(X))),
\end{align}

where $\mathcal{K}_{\pi_1}(X)$ is the subgroup of $\mathcal{E}(X)$ consisting of homotopy self-equivalences which induce the identity outer automorphism of the fundamental group.

We also note the following result.

**Lemma 3.1.** Let $X$ satisfy the conditions of Proposition 1.1, then we have that $\text{innaut} \pi_1$ is contained in the image of $\mathcal{E}^*(X) \to \text{aut} \pi_1$. 