1 Preordered and partially ordered rings

This work is devoted to the investigation of categories of partially ordered rings. As a first step, the categorial framework needs to be set up by introducing some of the categories that will be used throughout. The largest category that will ever occur is the category of preordered rings. The categories of partially ordered rings and reduced partially ordered rings are subcategories thereof. Some of the most basic properties of these categories will be discussed.

Preordered rings are rings with some additional structure. Therefore it is necessary to start with a brief discussion of the underlying rings. All rings are commutative and have a unit element. Most of the rings that we consider are reduced, i.e., without nilpotent elements. The rings together with their homomorphisms form a category denoted by $\text{R}$. The zero ring is included in $\text{R}$. The category $\text{R}$ is a concrete category or construct ([83], p. 26; [1], Definition 5.1; [60], Definition 12.7) via the underlying sets functor $U = U_{\text{SETS,R}} : \text{R} \to \text{SETS}$ to the category of sets. A functor $F : D \to C$ that is left adjoint to a forgetful functor is called a free functor ([1], Example 19.4(2)); with every $X \in ob(\text{D})$ it associates the free object over $X$. The forgetful functor $U_{\text{SETS,R}}$ has a left adjoint functor $F = F_{\text{R,SETS}}$; if $X \in ob(\text{SETS})$ then $F(X)$ is the polynomial ring $\mathbb{Z}[T_x ; x \in X]$. The category $\text{R}$ is wellpowered ([83], p. 126; [1], Definition 7.82; [60], Definition 6.27) and co-wellpowered ([83], loc. cit; [1], Definition 7.87, Example 7.90(2); [60], loc. cit.), it is strongly complete ([1], Definition 12.2, Proposition 12.5) and strongly cocomplete (as can be checked by using the dual of [60], Theorem 23.8).

Deviating slightly from the usage in [73], p. 140, Definition 1, and [77], p. 776, we define a preordering of the ring $A$ to be a subset $P$, called the positive cone, having the following properties:

\begin{align}
(a) & \quad P + P \subseteq P; \\
(b) & \quad P \cdot P \subseteq P; \\
(c) & \quad \forall x \in A : x^2 \in P; \\
(d) & \quad \text{supp}(P) = P \cap -P \subseteq A \text{ is an ideal.}
\end{align}

The ideal $\text{supp}(P)$ is called the support of $P$. A ring with a preordering is a preordered ring and is denoted by $(A, P)$ or $(A, A^+)$ some similar
symbol. Occasionally we also omit the positive cone from the notation if it is clear from the context that we are discussing a preordered ring. The preordered rings are the objects of a category which we denote by \( \text{PREOR} \). The morphisms from \((A, P)\) to \((B, Q)\) in this category are the ring homomorphisms \( f: A \to B \) with \( f(P) \subseteq Q \). There is a forgetful functor \( U_{\mathcal{R}, \text{PREOR}}: \text{PREOR} \to \mathcal{R} \). Every ring is the underlying ring of some preordered ring since, given a ring \( A \), the entire ring is always a preordering of \( A \).

A preordered ring \((A, P)\) is said to be **partially ordered** if \( \text{supp}(P) = \{0\} \) (cf. \cite{27}, p. 32). Such a ring will be called a **poring**. The positive cone \( P \) determines a partial order of \( A \) through: \( a \preceq b \) if and only if \( b - a \in P \). This partial order returns \( P \) as its set of positive elements. The partial order is **total** if \( P \cup -P = A \). The full subcategories of \( \text{PREOR} \) whose objects are the porings or the reduced porings are denoted by \( \text{POR} \) or \( \text{POR}/\mathcal{N} \), resp.

Via the underlying ring functors \( U_{\mathcal{R}, \text{POR}} \) and \( U_{\mathcal{R}, \text{POR}/\mathcal{N}} \), both \( \text{POR} \) and \( \text{POR}/\mathcal{N} \) are concrete categories over \( \mathcal{R} \). Through composition of forgetful functors they are also concrete over \( \text{SETS} \). Not every ring is the underlying ring of a poring or a reduced poring. To determine exactly which rings are, recall that there are various notions of reality for rings ([73], p. 103, Definition 1; \cite{77}, Definition 2.1). The ring \( A \) is **semireal** if \( 1 + a_1^2 + \ldots + a_n^2 \neq 0 \) for all \( n \in \mathbb{N} \) and all \( a_1, \ldots, a_n \in A \); it is **real** if \( a_1^2 + \ldots + a_n^2 = 0 \) always implies \( a_1 = \ldots = a_n = 0 \). We add another item to this list of notions: The ring \( A \) is **weakly real** if \( a_1^2 + \ldots + a_n^2 = 0 \) implies that \( a_1^2 = \ldots = a_n^2 = 0 \). It follows from the definitions and \cite{73}, p. 104, Satz 1, or \cite{77}, Lemma 2.9, that \( A \) is real if and only if it is weakly real and reduced. The ring \( \mathbb{R}[T]/[T^2] \) is weakly real and not reduced, hence it is not real. The zero ring is both real and weakly real, but it is not semireal. If the weakly real ring \( A \) is not the zero ring then it is semireal. On the other hand, \( \mathbb{R}[T_1, T_2]/(T_1^2 + T_2^2) \) is a semireal ring that is not weakly real. The full subcategories of \( \mathcal{R} \) whose objects are the semireal rings or the weakly real rings or the real rings are denoted by \( \text{SRR}, \text{WRR}, \text{and RR} \), resp. The different notions or reality are extended to ideals by saying that an ideal \( I \subseteq A \) is **semireal**, **weakly real**, or **real**, if the factor ring \( A/I \) has the respective property.

Suppose that \( A \) is a weakly real ring. Then the set

\[
P_w(A) = \{ a_1^2 + \ldots + a_n^2; \ n \in \mathbb{N}, a_i \in A \}\]

is a positive cone of \( A \); it will referred to as the **weak order**. In particular, \( A \) is the underlying ring of a poring. Conversely, let \((A, P)\) be a poring