It is a general experience in commutative algebra that von Neumann regular rings have properties that are particularly close to those of fields. Similarly, von Neumann regular f–rings are close to totally ordered fields. This has been recognized a long time ago, in particular by model theorists, and led to the results in [79]; [81]; [82], §6; [123], §4; [122]. In this section we undertake a systematic investigation of the monoreflective subcategories of \( \text{VNRFR} \). Our results will reaffirm the above mentioned general experience. Our strongest results are about \( H \)-closed monoreflectors of \( \text{VNRFR} \). These are completely determined by their restrictions to \( \text{TOF} \) (Theorem 17.6). The question which subcategories of \( \text{TOF} \) belong to some \( H \)-closed monoreflector of \( \text{VNRFR} \) is answered completely in Theorem 17.7. We start with a few useful alternative characterizations of von Neumann regular f–rings.

**Proposition 17.1** Let \( (A, P) \) be a reduced poring. The following properties of \( (A, P) \) are equivalent:

(a) \( (A, P) \) is a von Neumann regular f–ring.
(b) \( \sigma_{(A,P)} : (A, P) \to \sigma(A, P) \) is an isomorphism onto the subring

\[ \{ a \in \sigma(A, P); \forall \alpha \in Sper(A, P): a(\alpha) \in \kappa_{(A,P)}(\alpha) \}. \]

(c) \( A \) is von Neumann regular and the support map \( supp : Sper(A, P) \to Spec(A, P) \) is injective.

**Proof** (a) \( \Rightarrow \) (c) \( A \) is von Neumann regular by hypothesis. The support map is injective for all f–rings. – (c) \( \Rightarrow \) (b) By Proposition 4.4 the minimal prime ideals (for a von Neumann regular ring that is: all prime ideals) belong to the image of the support. So, the support is bijective. Both spectra are 0–dimensional. Therefore the support is a homeomorphism. Also, \( Sper(\sigma_{(A,P)}): Sper(\sigma(A, P)) \to Sper(A, P) \) is a homeomorphism (Proposition 7.9). We shall identify all these spectra. It is clear that \( \sigma_{(A,P)} \) maps into the ring described in (b). We only need to show that this ring is the exact image. So, pick \( a \in \sigma(A, P) \) with \( a(\alpha) \in \kappa_{(A,P)}(\alpha) \) for every \( \alpha \in Sper(A, P) \). Since \( \kappa_{(A,P)}(\alpha) = A/\alpha \) (by
von Neumann regularity) we find some $a_\alpha \in A$ with $a_\alpha(\alpha) = a(\alpha)$. The sets

$$C_\alpha = \{ \beta \in Sper(A, P); a_\alpha(\beta) = a(\beta) \}$$

are constructible (Lemma 7.6) and cover $Sper(A, P)$. By compactness there is a finite subcover, say $Sper(A, P) = C_{\alpha_1} \cup \ldots \cup C_{\alpha_r}$. Without loss of generality one may assume that the covering sets are pairwise disjoint. Since $A$ is von Neumann regular there is some $c \in A$ such that $c(\alpha) = a_{\alpha_i}(\alpha)$ for each $i$ and each $\alpha \in C_{\alpha_i}$. But then $\sigma_{(A,P)}(c) = a$. \(\blacksquare\)

**Proposition 17.2** Suppose that $(A, P)$ is a von Neumann regular $f$-ring. Let $f : A \to B$, $g : B \to \sigma(A, P)$ be monomorphisms of rings with $gf = \sigma_{(A,P)}$. If $B$ is equipped with the restriction, say $Q$, of the partial order of $\sigma(A, P)$ then $(B, Q)$ is a von Neumann regular $f$-ring and $f$ is a $\text{POR}/\text{N}$-epimorphism.

**Proof** First we show that $(B, Q)$ is an $f$-ring. It suffices to prove that for each $b \in B$ there is some $c \in B$ with $g(c) = g(b) \lor 0$: The set $C = \{ \alpha \in Sper(A, P); g(b)(\alpha) \geq 0 \}$ is constructible. Therefore the characteristic function $\chi_C$ belongs to $A$. The element $c = f(\chi_C)b \in B$ has the desired property, i.e., $B$ is an $f$-ring.

By Corollary 14.7, $f$ is an epimorphism in $\text{POR}/\text{N}$, hence $Sper(f)$ is injective (Theorem 5.2). Since $Sper(gf)$ is a homeomorphism it follows that $Sper(f)$ is also a homeomorphism. Because $(B, Q)$ is an $f$-ring, the support map $\text{supp} : Sper(B, Q) \to \text{Spec}(B)$ is a homeomorphism onto the image. If it is surjective then $\text{Spec}(B)$ is homeomorphic to $Sper(A, P)$, hence $B$ is reduced and has Krull dimension 0, which means that $B$ is von Neumann regular. So it remains to show that $\text{supp}$ is surjective.

First pick some minimal prime ideal $p \subseteq B$. Since $g$ is an extension of rings there is a minimal prime ideal $q \subseteq \sigma(A, P)$ with $p = g^{-1}(q)$. There is a unique $\gamma \in Sper(\sigma(A, P))$ with $q = \text{supp}(\gamma)$. But then $p = \text{supp}(g^{-1}(\gamma))$, as claimed. Now let $p \subseteq B$ be any prime ideal and pick a minimal prime ideal $p_0 \subseteq p$, say $p_0 = \text{supp}(\beta), \beta \in Sper(B, Q)$. Let $\alpha = f^{-1}(\beta)$, let $\gamma \in Sper(\sigma(A, P))$ be such that $g^{-1}(\gamma) = \beta$. Then the extension $\kappa_{(A,P)}(\alpha) \subseteq \kappa_{\sigma(A,P)}(\gamma) = \rho(\alpha)$ is an algebraic field extension and $B/p_0$ is an intermediate ring, hence is a field as well. The prime ideal $p/p_0 \subseteq B/p_0$ must be trivial, i.e., $p = p_0$ is a minimal prime ideal, hence