2. A vector-valued Hahn-Banach Theorem

2.1 Definition: a) Given a vector space \( E \) and an ordered cone \( C \), a mapping \( p : E \to C \) is called sublinear (resp. superlinear), if
\[
p(x+y) \leq p(x) + p(y) \quad \text{for all } x, y \in E \quad (\text{subadditively})
\]
\[
p(x+y) \geq p(x) + p(y) \quad \text{for all } x, y \in E \quad (\text{respectively}),
\]
and
\[
p(\lambda x) = \lambda p(x) \quad \text{for all } \lambda \in \mathbb{R}_+, \, x \in E \quad (\text{positive homogeneity}).
\]

b) Let \( K \) be a convex subset of a vector space \( E \) and let \( C \) be an ordered cone. A mapping \( f : K \to C \) is convex, if
\[
f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \text{for all } \lambda \in [0,1], \, x, y \in K.
\]
it is called concave, if
\[
f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y) \quad \text{for all } \lambda \in [0,1], \, x, y \in K.
\]

2.2 Lemma: Let \( E \) be a locally convex vector space, \( U \) a convex neighborhood of 0 in \( E \) and let \( C \) be a tight imbedding cone for some vector lattice \( F \). If \( k : U \to F \) is concave and satisfies \( k(0) \leq 0 \), then
\[
\tilde{k}(x) := \begin{cases} 
\sup \left\{ \frac{1}{\lambda} k(\lambda x) : \lambda \in \mathbb{R}_+, \, \lambda x \in U \right\} & \text{for } x \neq 0 \\
0 & \text{for } x = 0
\end{cases}
\]
is an element of \( F \) for each \( x \in E \). \( \tilde{k} \) is the smallest positively homogeneous function \( q : E \to C \) such that \( q|_U \geq k \) and \( \tilde{k} \) is superlinear. Moreover, if \( F \) is a topological vector lattice and if \( k \) is continuous at 0, the \( \tilde{k} \) is continuous.

**Proof:** Given \( x \in E \setminus \{0\} \), choose \( \epsilon > 0 \) such that \( [-\epsilon x, \epsilon x] \subset U \). If \( \lambda \in \mathbb{R}_+, \, \lambda > \epsilon \), is such that \( \lambda x \in U \) and if \( a := \min\left\{ \frac{\epsilon}{\lambda}, \frac{1}{2} \right\} \), we conclude from the concavity of \( k \):
\[
k(a\lambda x) = k(a(\lambda x) + (1-a)\cdot 0) \geq a \cdot k(\lambda x) + (1-a)k(0),
\]
whence
Furthermore, for \( \mu \in ]0, \varepsilon], \beta := \frac{\varepsilon}{\varepsilon + \mu} \), we obtain

\[
k(0) = k((\beta \mu + (1 - \beta)(-\varepsilon)) x) \geq \beta k(\mu x) + (1 - \beta) k(-\varepsilon x),
\]

which yields

\[
(2.2.2) \quad \frac{1}{\mu} \cdot k(\mu x) \leq \frac{1}{\mu B}(k(0) - (1 - \beta) k(-\varepsilon x)) \leq -\frac{1 - \beta}{\mu B} k(-\varepsilon x) = -\frac{1}{\varepsilon} k(-\varepsilon x).
\]

Inserting \( \mu := a \lambda \) in 2.2.2 and using (2.2.1) we deduce

\[
\frac{1}{\lambda} k(\lambda x) \leq -\frac{1}{\varepsilon} k(-\varepsilon x) - \frac{1-a}{a \lambda} k(0),
\]

and, observing the inequality

\[
\frac{1-a}{a \lambda} \leq \frac{1}{\varepsilon},
\]

we have

\[
(2.2.3) \quad \frac{1}{\lambda} k(\lambda x) \leq \frac{1}{\varepsilon} |k(-\varepsilon x)| + \frac{1}{\varepsilon} |k(0)|.
\]

Combining (2.2.2) and (2.2.3) we conclude

\[
(2.2.4) \quad \frac{1}{\lambda} k(\lambda x) \leq \frac{1}{\varepsilon} |k(-\varepsilon x)| + \frac{1}{\varepsilon} |k(0)|
\]

for every \( x \in \mathbb{R}_+^\mu \) such that \( \lambda x \in U \).

Therefore the set \( \{ \frac{1}{\lambda} k(\lambda x) : x \in \mathbb{R}_+^\mu, \lambda x \in U \} \) is bounded from above in \( F \),

which shows that \( \tilde{k}(x) \in F \).

Obviously, \( \tilde{k} \) is positively homogeneous. In order to prove the super-additivity, let \( u, v \in E \) be given.

For \( u = 0 \) or \( v = 0 \) the equality \( \tilde{k}(u + v) = \tilde{k}(u) + \tilde{k}(v) \) holds. We may therefore assume that \( u \neq 0, v \neq 0 \). If \( \lambda, \mu \in \mathbb{R}_+^\mu \) satisfy \( \lambda u \in U \) and \( \mu v \in U \) we deduce from the convexity of \( U \) that

\[
\frac{\mu \lambda}{\mu + \lambda} (u + v) = \frac{\mu}{\mu + \lambda} \lambda u + \frac{\lambda}{\mu + \lambda} \mu v \in U,
\]

hence

\[
k\left( \frac{\lambda \mu (u + v)}{\mu + \lambda} \right) \geq \frac{\mu}{\mu + \lambda} k(\lambda u) + \frac{\lambda}{\mu + \lambda} k(\mu v).
\]

Consequently,

\[
\tilde{k}(u + v) \geq \frac{\mu + \lambda}{\mu \lambda} k(-\frac{\mu \lambda}{\mu + \lambda} (u + v)) \geq \frac{1}{\lambda} k(\lambda u) + \frac{1}{\mu} k(\mu v).
\]

Passing to the respective suprema on the right side, we obtain

\[
\tilde{k}(u + v) \geq \tilde{k}(u) + \tilde{k}(v).
\]

Since \( \tilde{k}(x) \geq \frac{1}{\lambda} k(1 \cdot x) = k(x) \) for all \( x \in U \), \( \tilde{k} \) dominates \( k \) on \( U \). Finally, consider a positively homogeneous mapping \( q : E \to C \) satisfying \( q|_{U} \geq k \).