Chapter 1

Moduli spaces of PEL structures

In this chapter we introduce the basis for the moduli interpretation of most arithmetic quotients, in terms of abelian varieties with given polarization, endomorphism ring and level structure. This is Shimura’s theory of PEL structures, developed in [Sh2] and [Sh3] and summarized in [Sh4]. Although quite well-known, we present very briefly this relation, in order to fix notations for the rest of the book. In all later examples (from the next chapter on), all groups will be what is called “split over \( \mathbb{R} \)”, in which a maximal \( k \)-split torus is also a maximal \( \mathbb{R} \)-split one. However this is perhaps not so representative, so we also take some time in this chapter to consider a case which is, roughly speaking, the opposite of the split over \( \mathbb{R} \) type. This is the case of hyperbolic planes, which is considered in detail in [H2]. This case is included to give the reader the flavor of a more “typical” example of arithmetic quotients. Our later examples may in fact be very untypical, but nonetheless very beautiful, something which is also true of the hyperbolic planes.

1.1 Shimura’s construction

1.1.1 Endomorphism rings

Let \( V \) be an abelian variety over \( \mathbb{C} \), \( \text{End}(V) \) the endomorphism ring and

\[
\text{End}_\mathbb{Q}(V) = \text{End}(V) \otimes \mathbb{Q}
\]

the endomorphism algebra. A polarization, i.e., a linear equivalence class of ample divisors giving a projective embedding of \( V \), gives rise to a positive involution on \( \text{End}_\mathbb{Q}(V) \), the so-called Rosati involution:

\[
\varrho : \text{End}_\mathbb{Q}(V) \longrightarrow \text{End}_\mathbb{Q}(V) \\
\phi \mapsto \phi^\varrho.
\]
This can be defined, for example, by viewing the polarization as an isogeny $C : V \to V^\vee$, where $V^\vee$ is the dual abelian variety. The degree of this isogeny is the degree of the polarization. Then for any $\alpha \in \operatorname{End}_\mathbb{Q}(V)$, one sets $\alpha^e := C^{-1} \circ \alpha \circ C$ (see [Mi], §17 for a presentation along these lines). The Rosati involution gives rise to a positive definite bilinear form $(\alpha, \beta) \mapsto \operatorname{Tr}(\alpha \circ \beta^o)$ (loc. cit. 17.3).

If $A$ is a central simple algebra over $\mathbb{Q}$, an involution $^*$ on $A$ is called positive, if $\operatorname{tr}_{A|\mathbb{Q}}(x \cdot x^*) > 0$ for all $x \in A$, $x \neq 0$, where $\operatorname{tr}_{A|\mathbb{Q}}$ denotes the reduced trace (see Definition A.1.9). Assuming $(A, ^*)$ to be simple with positive involution, the $\mathbb{R}$-algebra $A(\mathbb{R})$ is isomorphic to one of the following (see [Sh2], Lemma 1)

(i) $M_r(\mathbb{R})$ with involution $x^* = {}^t X$;

(ii) $M_r(\mathbb{C})$ with involution $x^* = {}^t \overline{X}$, where $\overline{\cdot}$ is complex conjugation;

(iii) $M_r(\mathbb{H})$ with involution $x^* = {}^t \overline{X}$, where $\overline{\cdot}$ is quaternionic conjugation.

The algebras $A$ occurring in (i) and (iii) are central simple over $\mathbb{R}$, while those of (ii) are central simple over $\mathbb{C}$. The $\mathbb{Q}$-algebra $A$ itself is a $\mathbb{Q}$-form of one of these. The central simple algebras $A$ over $\mathbb{Q}$ are known to be the $M_n(D)$, where $D$ is a division algebra over $\mathbb{Q}$ (see section A.1.2). If the algebra $A$ has a positive involution, the same holds for $D$. The division algebras $D$ which can occur are also known.

**Proposition 1.1.1** Let $D$ be a division algebra over $\mathbb{Q}$ with a positive involution. Then $D$ occurs in one of the following cases:

I. A totally real algebraic number field $k$;

II. $D$ a totally indefinite quaternion algebra over $k$;

III. $D$ a totally definite quaternion algebra over $k$;

IV. $D$ is central simple with a $K|k$ involution of the second kind (see Definition A.1.3), where $K$ is an imaginary quadratic extension of $k$.

In case III the canonical involution on $D$ is the unique positive involution, while in case II the positive involutions correspond to $x \in D$ such that $x^2$ is totally negative in $k$. If the algebra $D$ has an involution of the second kind (see Theorem A.1.14) it is easy to see that it admits a positive one. It follows from the fact that $\operatorname{End}_\mathbb{Q}(V)$ is a semisimple algebra over $\mathbb{Q}$ with a positive involution that each simple factor is a total matrix algebra $M_n(D)$, with $D$ as in the proposition.

### 1.1.2 Abelian varieties with given endomorphisms

Let $(A, *)$ be a semisimple algebra over $\mathbb{Q}$ with positive involution, and let

$$\Phi : A \to GL(n, \mathbb{C})$$

(1.2)