Chapter 6

A Gem of the modular universe

In this chapter we will introduce a final variety whose existence derives from the set of 27 lines on a cubic surface – the invariant quintic fourfold $\mathcal{I}_5 \subset \mathbb{P}^5$, the unique invariant hypersurface of degree 5 under the natural action of $\text{Aut}(\mathcal{C}) \cong W(E_6)$ on $\mathbb{P}^5$. It is also probably a modular variety, and in fact a (very) Janus-like variety, but the proof of this fact remains to be completed. At any rate it is full of modular subvarieties, i.e., subvarieties which are modular varieties, and as an algebraic variety it has the following very special properties.

- $\mathcal{I}_5$ is the unique invariant of degree $d = 5$ for the reflection group $W(E_6)$ acting on $\mathbb{P}^5$. Hence the symmetry group of $\mathcal{I}_5$ is $W(E_6)$.

- The singular locus of $\mathcal{I}_5$ consists of a set of 120 double lines which meet ten at a time in 36 triple points.

- The 36 triple points correspond in a $W(E_6)$-equivariant manner with the 36 double sixes, the 120 double lines correspond in the same equivariant manner with the 120 trihedral pairs, of the set of 27 lines on a smooth cubic surface.

- There are 27 special hyperplane sections of $\mathcal{I}_5$ which split into a pentahedron (five $\mathbb{P}^3$'s) in that hyperplane (a $\mathbb{P}^4$). These correspond in an equivariant manner with the 27 lines.

- There are 45 $\mathbb{P}^3$'s which are contained in $\mathcal{I}_5$; these correspond in an equivariant manner with the 45 tritangents.

- Each of the 27 hyperplane sections contains five of the 45 $\mathbb{P}^3$'s, corresponding to the five tritangents containing a given line; each of the 45 $\mathbb{P}^3$'s is contained in three of the 27 hyperplanes, corresponding to the three lines which lie in a given tritangent.
6.1. THE WEYL GROUP $W(E_6)$

- The Hessian variety of $\mathcal{I}_5$, a degree $d = 18$ hypersurface in $\mathbb{P}^5$, intersects $\mathcal{I}_5$ in the union of the 45 $\mathbb{P}^3$'s – each with multiplicity 2.
- $\mathcal{I}_5$ is a self-Steinerian variety.
- There are 36 hyperplane sections which are a quintic threefold which is isomorphic to the Hessian variety $\text{Hess}(S_3)$ of the Segre cubic.
- The 36 singular points are resolved by cubic hypersurfaces in $\mathbb{P}^4$ which are all copies of the Segre cubic $S_3$.

As in the case of the Burkhardt quartic one may consider the configuration of the 36 points, which contains a lot of geometry, but generally speaking the intersections with $\mathcal{I}_5$ are not as interesting as was the case for $B_4$. The most special such space section which is an irreducible quintic is depicted in the frontispiece.

Since the dimension of the moduli space of cubic surfaces is four and $\mathcal{I}_5$ is so intricately related with the 27 lines, it is natural to inquire about a more immediate relation. For example, is the quintic $\mathcal{I}_5$ the moduli space of marked cubic surfaces? This turns out to not quite be true, and the relationship will be explicitly described below. This involves in particular the Coble variety $\mathcal{V}$, giving a surprising relation of $\mathcal{I}_5$ with moduli spaces of certain K3 surfaces.

6.1 The Weyl group $W(E_6)$

In this section we recall some well-known facts on the root system of type $E_6$, and describe also the arrangement and the configuration defined by the 36 reflection hyperplanes and 36 dual points, respectively. For the roots, etc., we adhere to the notations of Bourbaki.

6.1.1 Notations

We use the same notation as above for the 27 lines on a cubic surface in $\mathbb{P}^3$: $a_1, ... a_6, b_1, ..., b_6, c_{12}, ..., c_{56}$. The 36 double sixes are:

$$N = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}, \quad (1)$$

$$N_{ij} = \begin{bmatrix} a_i & b_i & c_{jk} & c_{ji} & c_{jm} & c_{jn} \\ a_j & b_j & c_{ik} & c_{il} & c_{im} & c_{in} \end{bmatrix}, \quad (15)$$

$$N_{lmn}^{-1} = \begin{bmatrix} a_i & a_j & a_k & c_{mn} & c_{ln} & c_{lm} \\ c_{jk} & c_{ik} & c_{ij} & b_l & b_m & b_n \end{bmatrix} \quad (20).$$

The 45 tritangents are:

$$ (ij) = < a_i b_j c_{ij}>, \quad i \neq j \quad (30)$$

$$ (ij, kl, mn) = < c_{ij} c_{kl} c_{mn}> \quad (15).$$

\textsuperscript{1}here we switch notations from $N_{ijk}$ in equation (4.1) to $N_{lmn}$ for convenience