An Asymptotic Evaluation of Heat Kernel for Short Time

In Honor of P.A. Meyer and J. Neveu

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Consider the following heat equation

$$\frac{\partial u}{\partial t} = (\Delta + V)u,$$

(1)

where $\Delta$ is the Laplacian operator on $\mathbb{R}^d$ and $V$ is a continuous function on $\mathbb{R}^d$. Under mild assumptions on $V$ the fundamental solution of equation (1) exists and can be expressed by the Feynman-Kac formula (cf.[2]). This fundamental solution is called the heat kernel.

The purpose of this paper is to prove the following theorem, which gives an asymptotic evaluation of the heat kernel for short time.

**Theorem.** Let $V$ be a continuous function on $\mathbb{R}^d$. Assume there exist positive constants $C, C_1$ and $C_2$ such that

$$V(x)^+ \leq C(1 + |x|^2),$$

(2)

$$V(x)^- \leq C_1 e^{C_2 |x|^2}.$$  

(3)

Let $q(t, x, y)$ be the fundamental solution of the heat equation (1). Then we have

$$\lim_{t \to 0} \frac{1}{t} \log \frac{q(t, x, y)}{p(t, x, y)} = \int_0^1 V((1 - s)x + sy) \, ds,$$

(4)

where $p(t, x, y)$ is the transition density of a standard Brownian motion.

The main tool for proving this theorem is the Feynman-Kac formula. We recall it for the reader's convenience.

Let $\Omega = C \left([0, \infty), \mathbb{R}^d\right)$ be the collection of all continuous functions from $[0, \infty)$ to $\mathbb{R}^d$. For $\omega \in \Omega$, let $X_t(\omega) = \omega(t)$. Let $\mathcal{F}_t = \sigma\{X_s, s \leq t\}, \mathcal{F} = \sigma\{X_s, s < \infty\}$. We denote by $(\mathcal{P}_x, x \in \mathbb{R}^d)$ the unique family of probability measures on $(\Omega, \mathcal{F})$ such that $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathcal{P}_x)$ is a standard Brownian motion. Let $y \in \mathbb{R}^d$ and $t > 0$. Put

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\[ Y_s(\omega) = X_s(\omega) - \frac{s^t}{t}(X_t(\omega) - y), \quad s \geq 0. \]

Then under \( P_x \) the process \( (Y_s, 0 \leq s \leq t) \) is a Brownian bridge from \( x \) to \( y \) on \([0, t]\)

and \( (Y_s, t \leq s < \infty) \) is a Brownian motion with \( Y_t = y \). Moreover, under \( P_x \) these two processes are independent. We denote by \( P_{x,y,t} \) the distribution of the process \( (Y_s, s \geq 0) \) on \((\Omega, F)\) under \( P_x \). We call \( P_{x,y,t} \) the \((0, x; t, y)\)-Brownian bridge measure.

Under mild assumptions on \( V \) it was shown that the heat kernel for (1) can be expressed

by the following Feynman-Kac formula (cf. [2, Theorem 3.2]):

\[
q(t, x, y) = p(t, x, y)E_{x,y,t}[e^{\int_0^t V(X_s) \, ds}]
= p(t, x, y)E_0[e^{\int_0^t V(x + \frac{t}{2}(y-x) + X_s - \frac{t}{2}X_t) \, ds}]
= p(t, x, y)E_0[e^{\int_0^t V(x + s(y-x) + \sqrt{t}(x_s - sX_t)) \, ds}],
\]

where

\[
p(t, x, y) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x - y|^2}{2t}}.
\]

We are going to prove the theorem. To begin with we prepare a lemma.

**Lemma.** Let \( \{\xi(\varepsilon), \varepsilon > 0\} \) be a family of integrable random variables such that

\[
\lim_{\varepsilon \downarrow 0} E[\xi(\varepsilon)] \text{ exists and is finite. If}
\]

\[
\lim_{\varepsilon \downarrow 0} E[\xi(\varepsilon)(e^{\varepsilon \xi(\varepsilon)} - 1)] = 0,
\]

then we have

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \log E[e^{\varepsilon \xi(\varepsilon)}] = \lim_{\varepsilon \downarrow 0} E[\xi(\varepsilon)].
\]

**Proof.** Since \( 1 + x \leq e^x \leq 1 + x + x(e^x - 1) \), we have

\[
\xi(\varepsilon) \leq \frac{1}{\varepsilon} [e^{\varepsilon \xi(\varepsilon)} - 1] \leq \xi(\varepsilon) + \xi(\varepsilon)(e^{\varepsilon \xi(\varepsilon)} - 1).
\]

This together with (6) imply

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (E[e^{\varepsilon \xi(\varepsilon)}] - 1) = \lim_{\varepsilon \downarrow 0} E[\xi(\varepsilon)],
\]

which is equivalent to (7).

**Corollary 1.** If instead of (6) we assume

\[
\lim_{\varepsilon \downarrow 0} E[\xi(\varepsilon)^2(e^{\varepsilon \xi(\varepsilon)} + 1)] = 0,
\]

then (7) holds.