CONTINUOUS MAASSEN KERNELS AND THE INVERSE OSCILLATOR

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Dedicated to P.A. Meyer to his 60th birthday

Summary: The quantum stochastic differential equation of the inverse oscillator in a heat bath of oscillators is solved by the means of a calculus of continuous and differentiable Maassen kernels. It is shown that the time development operator does not only map the Hilbert space of the problem into itself, but also vectors with finite moments into vectors with finite moments. The vacuum expectation of the occupancy numbers coincides for pyramidal ordered times with a classical Markovian birth process showing the avalanche character of the quantum process.

§ 0. Introduction

The quantum mechanical oscillator has the Hamiltonian $\omega_0 b^+ b$, where $b$ and $b^+$ are the usual annihilation and creation operators. The inverse oscillator has the Hamiltonian $-\omega_0 b^+ b$. Coupled to a heat bath the inverse oscillator has the Hamiltonian

$$-\omega_0 b^+ b + \sum_{\lambda \in \Lambda} (\omega_0 + \omega_\lambda) a^\lambda_+ a_\lambda + \sum_{\lambda \in \Lambda} (g_\lambda a_\lambda b + g_\lambda^* a^\lambda_+ b^*).$$

As this Hamiltonian is not bounded below it cannot describe a real physical system; it can be used, however, to approximate the initial behavior of real physical systems, e.g. in the case of superradiance, as it is shown in § II.1 [2], [3], [11].

Using the interaction representation and singular coupling limit we arrive to the quantum stochastic differential equation for the time development operator

$$dU_{t,s} = (-ibd_\lambda^t - ib^+d^\lambda_+ - \frac{1}{2} bb^+dt)U_{t,s}.$$ 

This is a well-known equation, already mentioned in one of the early papers of Hudson and Parthasarathy [5].

The mathematical problem is that the coefficients $b$ and $b^+$ are unbounded operators. We treat it in considering the matrix elements

$$\langle m | U_{t,s} | n \rangle$$

as Maassen kernels. Here again a problem arises as the kernels are not bounded in the Maassen sense. Due to the simple algebraic structure, however, all convolutions of these kernels are allowed.
In chapter I we reconstruct the theory of Maassen kernels without the exponential bond used by Maassen. We introduce continuity and differentiability in a slightly different way and obtain an elementary theory which uses only calculus and Lebesgue integrations. We regain Maassen's theorem connecting differentiation and integration similar to the fundamental theorem of calculus and Maassen's and Robinson's general Itô-formula [7], [8], [9], [10], [12]. From there one can obtain several Itô tables for adapted processes. We have the usual Itô table for forward adapted processes

\[
\begin{align*}
\text{(3)} & \quad \begin{bmatrix}
da & da^*_t \\
da & 0 \\
da^*_t & 0
\end{bmatrix} \\
\text{For backward adapted processes we obtain} & \\
\text{(4)} & \quad \begin{bmatrix}
da & da^*_t \\
da & 0 \\
da^*_t & 0
\end{bmatrix}
\end{align*}
\]

and if one of the processes is forward adapted and the other backward adapted we have

\[
\begin{align*}
\text{(5)} & \quad \begin{bmatrix}
da_t & da^*_t \\
da & 0 \\
da^*_t & 0
\end{bmatrix}
\end{align*}
\]

In chapter II we investigate the special structure of the inverse oscillator in a bath. Due to the quadratic Hamiltonian the Heisenberg equations are linear and can be solved easily. In II.2 we calculate the Heisenberg equations going back to the finite heat bath and performing the singular coupling limit. We obtain

\[
\begin{align*}
\text{(6)} & \quad b^*_{t,s} = U^+_{t,s} b^+ U_{t,s} = e^{(t-s)/2} b^+ + \int_s^t e^{(t-t')/2} \, da_t.
\end{align*}
\]

We see that for \( t \to \infty \)

\[
\begin{align*}
\text{(7)} & \quad e^{-t/2} b^*_{t,0} \to b^+ + i \int_0^t e^{+t/2} \, da_t = B^+.
\end{align*}
\]

As \( B \) and \( B^+ \) commute we can interpret them as classical quantities which might be understood as the macroscopic quantities after amplification [2]. Assume for \( t = 0 \) as statistical operator the vacuum for the bath and the density matrix \( \rho \) for the \( b \) and \( b^+ \), then \( B \) and \( B^+ \) are distributed with respect to the classical probability law given by a smeared out Wigner transform of \( \rho \).