\begin{align*}
c_n \left( \frac{1}{2} \right)^n \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( n+\frac{1}{2} \right)} n! \int_{-1}^{1} (1-t^2)^{n+1} \frac{dt}{8^k} \\
= c_n \left( \frac{1}{2} \right)^n \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( n+\frac{1}{2} \right)} n! \int_{0}^{1} (1-t^2)^{\frac{n+q-3}{2}} t^{\frac{n}{2}} dt \\
= c_n n! \left( \frac{1}{2} \right)^n \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( n+\frac{1}{2} \right)} \cdot \frac{\Gamma \left( \frac{n+q-3}{2} \right) \cdot \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( n+\frac{1}{2} \right)}.
\end{align*}

Therefore
\[ P_n(t) = \frac{\omega_q}{\omega_{q-1}} \cdot \frac{1}{\sqrt{N(q,n)}} \cdot \frac{\Gamma \left( n+\frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right)} \cdot \frac{2^n}{n!} t^n + \ldots. \]

By (3)
\[ \frac{\omega_q}{\omega_{q-1}} = \frac{\sqrt{N(q,n)}}{\Gamma \left( \frac{1}{2} \right)} \]
so that
\[ (27) \quad P_n(t) = \frac{1}{\sqrt{N(q,n)}} \cdot \frac{\Gamma \left( n+\frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right)} \cdot \frac{2^n}{n!} t^n + \ldots. \]

**FUNK - HECKE FORMULA**

Before going further into the details of the Legendre functions we shall discuss a formula which will prove to be the basis of a great many special results.

Let us consider an integral of the form
\[ F(\alpha, \beta) = \int_{\Omega_q} f(\alpha, \eta) P_n(\beta, \eta) d\omega_q(\eta) \]

where \( f(t) \) is a continuous function for \(-1 \leq t \leq 1\) and the integration is carried out with respect to \( \eta \). Then with any orthogonal matrix \( A \)
\[ F(\mathbf{A}^\alpha, \mathbf{A}^\beta) = \int_{\Omega_q} f(\mathbf{A}^\alpha \cdot \eta) P_n(\mathbf{A}^\beta \cdot \eta) \, d\omega_q(\eta) \]

\[ = \int_{\Omega_q} f(\alpha \cdot \mathbf{A}^\ast \eta) P_n(\beta \cdot \mathbf{A}^\ast \eta) \, d\omega_q(\eta) \]

where \( \mathbf{A}^\ast \) is the adjoint (transpose) of \( \mathbf{A} \). Now the surface elements \( d\omega_q(\mathbf{A}^\ast \eta) \) and \( d\omega_q(\eta) \) are equal so that (28) becomes

\[ F(\mathbf{A}^\alpha, \mathbf{A}^\beta) = \int_{\Omega_q} f(\alpha \cdot \mathbf{A}^\ast \eta) P_n(\beta \cdot \mathbf{A}^\ast \eta) \, d\omega_q(\mathbf{A}^\ast \eta) \]

This is equal to \( F(\alpha, \beta) \) because we may regard \( \mathbf{A}^\ast \eta \) as the new variables. Using the same argument now which led to Lemma 5, we see that \( F(\alpha, \beta) \) is a function of the scalar product only, which gives us

\[ \int_{\Omega_q} f(\alpha \cdot \eta) P_n(\beta \cdot \eta) \, d\omega_q(\eta) = \Phi(\alpha, \beta) \]

Now as a function of \( \beta \) this is a spherical harmonic of degree \( n \). As it depends on the scalar product only, it has the same symmetry which characterizes \( P_n(\alpha \cdot \beta) \). Therefore we get

\[ \int_{\Omega_q} f(\alpha \cdot \eta) P_n(\beta \cdot \eta) \, d\omega_q(\eta) = \lambda P_n(\alpha \cdot \beta) \]

In order to determine \( \lambda \) set \( \alpha = \beta = \varepsilon_q \) and

\[ \eta = \eta_q = t \varepsilon_q + \sqrt{1-\varepsilon^2} \, \gamma_{q-1} \]

Then with

\[ d\omega_q = (4-\varepsilon^2)^{\frac{q-3}{2}} \, d\omega_{q-1} \]

we get

\[ \lambda = \lambda P_n(1) = \int_{\Omega_{q-1}} \int_{-1}^{+1} f(t) P_n(t)(4-\varepsilon^2)^{\frac{q-3}{2}} \, dt \, d\omega_{q-1} \]

\[ = \omega_{q-1} \int_{-1}^{+1} f(t) P_n(t)(4-\varepsilon^2)^{\frac{q-3}{2}} \, dt \]