4. Main results on $S_{g,1}$ (the principally polarized case)

In this chapter we collect the main propositions about the supersingular locus $S_{g,1} \subset A_{g,1} \otimes \mathbb{F}_p$ which will be proved separately in Chapter 6, Chapter 7 and in Chapter 8, and we prove the main theorem on the structure of $S_{g,1}$ using these propositions. We also explain some strategical techniques which we will use in the proofs of the propositions.

4.1. Existence of rigid PFTQ for a principally polarized supersingular abelian variety.

**Proposition.** Let $(X, \lambda)$ be a principally polarized supersingular abelian variety of dimension $g$ over $k$. Then there is a rigid PFTQ over $k$ (with respect to some $\eta$ over $k$) \{$(Y_i, (0 \leq i \leq g-1)); \rho_i (1 \leq i \leq g-1)$\} such that $(Y_0, \eta_0) \cong (X, \lambda)$. Furthermore, there are only a finite number of such PFTQs up to isomorphism.

This is an immediate consequence of Proposition 6.3.

4.2. A parameter space of principally polarized supersingular abelian varieties.

Two polarizations $\mu$ and $\mu'$ of an abelian variety $X$ are called equivalent if there is an automorphism $\phi$ of $X$ such that $\phi' \circ \mu \circ \phi = \mu'$.

Let $K = \mathbb{F}_p$. Let $\Lambda$ be a set of representatives of equivalence classes of polarizations $\eta$ of $E^g \otimes K$ satisfying (3.6.1). Then we have a canonical morphism

$$\Psi : \bigsqcup_{\eta \in \Lambda} \mathcal{P}_{g, \eta} \to S_{g,1} \otimes \mathbb{F}_p,$$

where $S_{g,1}$ is the supersingular locus in $A_{g,1}$ (see 1.10). Note that $\Psi$ is defined over $\mathbb{F}_p$ by 3.6 and Lemma 3.7. From Proposition 4.1 we get

**Corollary.** The morphism $\Psi$ is quasi-finite and surjective.

4.3. The important properties of $\mathcal{P}_g'$.

**Proposition.** Let \{\(G_{g-1} \subset ... \subset G_0\)\} be the universal IFTQ of group schemes over $\mathcal{P}_g$ (see Corollary 3.10).

i) $\mathcal{P}_g'$ is non-singular and geometrically integral of dimension $[g^2/4]$.

ii) The generic fiber of $G_0$ over $\mathcal{P}_g'$ has a-number equal to 1.

In Chapter 7 we give a proof of the "Weak Form of Proposition 4.3", i.e. replacing i) by
i') \( \mathcal{P}_g' \) is geometrically irreducible of dimension \([g^2/4]\).

And we sketch a proof of the fact that \( \mathcal{P}_g' \) is non-singular in 11.3; We also give a complete proof of this fact for \( g = 4 \) in 9.7.

4.4. The important properties of \( \mathcal{P}_{g, \eta}' \).

**Corollary.** Let \( \eta \) be a polarization of \( E^2 \otimes K \) satisfying (3.6.1). Let \( \{ Y_i \ (0 \leq i < g); \rho_i \ (0 < i < g) \} \) be the universal PFTQ over \( \mathcal{P}_{g, \eta}' \) (see Lemma 3.7).

i) \( \mathcal{P}_{g, \eta}' \) is non-singular and geometrically integral of dimension \([g^2/4]\).

ii) The generic fiber of \( Y_0 \) over \( \mathcal{P}_{g, \eta}' \) is supergeneral.

4.5. About the structure of \( \mathcal{P}_g \).

**Remark.** Note that \( \mathcal{P}_g \) is integral and non-singular for \( g \leq 3 \) (see Examples 3.8 and Example 9.4). Note also that the subscheme \( \mathcal{P}_g' \) is integral and non-singular for any \( g \). However, in general \( \mathcal{P}_g \) is neither non-singular nor irreducible (see Example 9.6).

4.6. Some class numbers, \( H_g(p, 1) \).

Let \( B \) be the definite quaternion algebra over \( \mathbb{Q} \) with discriminant \( p \) (as in (1.2.5)). Let \( O \) be a maximal order of \( B \) (as in (1.2.4)). By a theorem of Eichler (cf. [92, Lemma 4.4]), every left \( O \)-lattice in \( B^\otimes g \) is equal to \( O^\otimes x \) for some \( x \in GL_g(B) \). Let

\[
G = \{ h \in M_g(B)| h \tilde{h}^t = rI \text{ for some } r \in \mathbb{Q}^\times \}. \tag{4.6.1}
\]

Two \( O \)-lattices \( L \) and \( L' \) in \( B^\otimes g \) are called globally equivalent (denoted by \( L \sim L' \)) if there exists \( h \in G \) such that \( L' = Lh \).

Let

\[
\Sigma := \{ f \in M_g(O)| f = (\tilde{f})^t \text{ is positive definite} \}. \tag{4.6.2}
\]

Two elements \( f, f' \in \Sigma \) are called quasi-equivalent (denoted by \( f \sim f' \)) if there exists \( \gamma \in GL_g(O) \) and a positive rational number \( m \) such that \( \gamma^t f \gamma = mf' \), and \( f \) and \( f' \) are called equivalent (denoted by \( f \approx f' \)) if in addition we have \( m = 1 \).

By the argument of [31, Lemma 2.5], the map \( x \mapsto x \bar{x}^t \) induces a one-to-one correspondence between the global equivalence classes of left \( O \)-lattices in \( B^\otimes g \) and \( \Sigma/\sim \).

Let

\[
N_p = O_p^\otimes \begin{pmatrix} 1 - r & 0 \\ 0 & \pi 1_r \end{pmatrix} \xi, \tag{4.6.3}
\]

where \( \pi \) is a prime element of \( O_p \) and \( \xi \in GL_g(B_p) \) such that

\[
\xi \xi^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.6.4}
\]

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