4. Main results on $S_{g,1}$
(the principally polarized case)

In this chapter we collect the main propositions about the supersingular locus $S_{g,1} \subset A_{g,1} \otimes F_p$ which will be proved separately in Chapter 6, Chapter 7 and in Chapter 8, and we prove the main theorem on the structure of $S_{g,1}$ using these propositions. We also explain some strategical techniques which we will use in the proofs of the propositions.

4.1. Existence of rigid PFTQ for a principally polarized supersingular abelian variety.

**Proposition.** Let $(X, \lambda)$ be a principally polarized supersingular abelian variety of dimension $g$ over $k$. Then there is a rigid PFTQ over $k$ (with respect to some $\eta$ over $k$) $\{Y_i (0 \leq i \leq g - 1) ; \rho_i (1 \leq i \leq g - 1)\}$ such that $(Y_0, \eta_0) \cong (X, \lambda)$. Furthermore, there are only a finite number of such PFTQs up to isomorphism.

This is an immediate consequence of Proposition 6.3.

4.2. A parameter space of principally polarized supersingular abelian varieties.

Two polarizations $\mu$ and $\mu'$ of an abelian variety $X$ are called equivalent if there is an automorphism $\phi$ of $X$ such that $\phi' \circ \mu \circ \phi = \mu'$.

Let $K = F_p$. Let $\Lambda$ be a set of representatives of equivalence classes of polarizations $\eta$ of $E^g \otimes K$ satisfying (3.6.1). Then we have a canonical morphism

$$\Psi : \bigsqcup_{\eta \in \Lambda} \mathcal{P}_{g, \eta} \to S_{g,1} \otimes F_p,$$

(4.2.1)

where $S_{g,1}$ is the supersingular locus in $A_{g,1}$ (see 1.10). Note that $\Psi$ is defined over $F_p^2$ by 3.6 and Lemma 3.7. From Proposition 4.1 we get

**Corollary.** The morphism $\Psi$ is quasi-finite and surjective.

4.3. The important properties of $\mathcal{P}_g'$.

**Proposition.** Let $\{G_{g-1} \subset ... \subset G_0\}$ be the universal IFTQ of group schemes over $\mathcal{P}_g$ (see Corollary 3.10).

i) $\mathcal{P}_g'$ is non-singular and geometrically integral of dimension $[g^2/4]$.

ii) The generic fiber of $G_0$ over $\mathcal{P}_g'$ has a-number equal to 1.

In Chapter 7 we give a proof of the "Weak Form of Proposition 4.3", i.e. replacing i) by
4.4. The important properties of $\mathcal{P}_{g, \eta}'$.

**Corollary.** Let $\eta$ be a polarization of $E^g \otimes K$ satisfying (3.6.1). Let $\{Y_i \mid 0 \leq i < g; \rho_i \mid 0 < i < g\}$ be the universal PFTQ over $\mathcal{P}_{g, \eta}'$ (see Lemma 3.7).

i) $\mathcal{P}_{g, \eta}'$ is non-singular and geometrically integral of dimension $[g^2/4]$.

ii) The generic fiber of $Y_0$ over $\mathcal{P}_{g, \eta}'$ is supergeneral.

4.5. About the structure of $\mathcal{P}_g$.

**Remark.** Note that $\mathcal{P}_g$ is integral and non-singular for $g \leq 3$ (see Examples 3.8 and Example 9.4). Note also that the subscheme $\mathcal{P}_g'$ is integral and non-singular for any $g$. However, in general $\mathcal{P}_g$ is neither non-singular nor irreducible (see Example 9.6).

4.6. Some class numbers, $H_g(p, 1)$.

Let $B$ be the definite quaternion algebra over $\mathbb{Q}$ with discriminant $p$ (as in (1.2.5)). Let $O$ be a maximal order of $B$ (as in (1.2.4)). By a theorem of Eichler (cf. [92, Lemma 4.4]), every left $O$-lattice in $B^{O_g}$ is equal to $O^{O_g}x$ for some $x \in GL_g(B)$. Let

$$G = \{h \in M_g(B) \mid hh^t = rI \text{ for some } r \in \mathbb{Q}^\times\}. \quad (4.6.1)$$

Two $O$-lattices $L$ and $L'$ in $B^{O_g}$ are called globally equivalent (denoted by $L \sim L'$) if there exists $h \in G$ such that $L' = Lh$.

Let

$$\Sigma := \{f \in M_g(O) \mid f = (\tilde{f})^t \text{ is positive definite}\}. \quad (4.6.2)$$

Two elements $f, f' \in \Sigma$ are called quasi-equivalent (denoted by $f \sim f'$) if there exists $\gamma \in GL_g(O)$ and a positive rational number $m$ such that $\gamma^t f \gamma = mf'$, and $f$ and $f'$ are called equivalent (denoted by $f \approx f'$) if in addition we have $m = 1$.

By the argument of [31, Lemma 2.5], the map $x \mapsto x\tilde{x}^t$ induces a one-to-one correspondence between the global equivalence classes of left $O$-lattices in $B^{O_g}$ and $\Sigma/\sim$.

Let

$$N_p = O_{p}^{O_g} \begin{pmatrix} 1_{g-r} & 0 \\ 0 & \pi_1 r \end{pmatrix} \xi, \quad (4.6.3)$$

where $\pi$ is a prime element of $O_p$ and $\xi \in GL_g(B_p)$ such that

$$\xi \xi^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4.6.4)$$