A NOTE ON STRONG, NON-ANTICIPATING SOLUTIONS
FOR STOCHASTIC DIFFERENTIAL EQUATIONS:
WHEN IS PATH-WISE UNIQUENESS NECESSARY?

Deborah Allinger

Abstract. A necessary and sufficient condition for obtaining strong, non-anticipating solutions is given. As a corollary, we show that path-wise uniqueness is necessary for the existence of strong solutions in a large class of stochastic differential equations.

1. Introduction. Let $\Omega, F, P$ be a certain probability space, $(\mathcal{L}_t), 0 \leq t \leq 1,$ a non-decreasing family of sub-$\sigma$-algebras of $F,$ and $W = (W_t, \mathcal{L}_t)$ a Wiener process. Denote by $(C_1, B_1)$ the measurable space of continuous functions $x = (x_t)$ on $[0,1]$ with the $\sigma$-algebra $B_1 = \sigma\{x: x_s, s \leq t\}.$ Also, set $B_t = \sigma\{x: x_s, s \leq t\}.$ Let $\alpha(t,x)$ be a measurable, non-anticipative (i.e., $B_t$-measurable for each $t$) real-valued functional. We say that the $P$-a.s. continuous random process $\xi = (\xi_t)$ is a strong solution of the stochastic differential equation (s.d.e.).

\begin{equation}
\frac{d\xi_t}{dt} = \alpha(t,\xi)dt + dW_t
\end{equation}

if, for each $t,$ the variables $\xi_t$ are $\mathcal{L}_t$-measurable,

$$P\left(\int_0^1 |\alpha(t,\xi)| dt < \infty \right) = 1,$$

and with probability 1 for each $t,$

\begin{equation}
\xi_t = \int_0^t \alpha(s,\xi)ds + W_t.
\end{equation}
In particular, whenever \( F_t = F_t^W, \ 0 \leq t \leq 1 \), a strong solution \( \xi_t \) takes the form

\[
\xi_t(\omega) = \phi(t, W(\omega)) = \langle \delta_t, f(W(\omega)) \rangle, \quad (\lambda \times \text{P-a.s.}),
\]

where \( \lambda \) is Lebesgue measure on \([0,1]\). Here \( \delta_t \) represents evaluation of \( f(W(\omega)) \) at time \( t \); \( f \) is a transformation from \( C_1 \) into \( C_1 \). Letting \( F: C_1 \rightarrow C_1 \) denote the transformation which takes \( x \in C_1 \) to the function \( x - \int_0^t \alpha(s,x)ds \), we can rewrite (0.2) as

\[
\xi_t(\omega)(t) = W_t(\omega), \quad (\omega-\text{a.s.}),
\]

and observe that (0.2), (0.3) are inverse expressions. In other words, existence of a strong solution for the s.d.e. in (0.2) also shows that the non-linear operator, \( F \), can be causally inverted in the sense that there is a transformation, \( f \), which for each \( t \), satisfies

\[
f^{-1}(B_t) \subseteq B_t
\]

and such that

\[
(0.4) \quad F(f(y)) = y
\]

for \( y \) in a set of Wiener measure one. By substituting (0.3) into (0.2) we can represent \( f \) explicitly as

\[
(0.5) \quad f(y)(t) = y(t) + \int_0^t \phi(s,y)ds
\]

where \( \phi(s,y) = \alpha(s,f(y)) \), \((\lambda \times \nu \text{-a.s.})\). \( \nu \) is Wiener measure on \( C_1 \). Moreover, in case the resulting strong solution determines a distribution, \( \mu_{\xi} \), on \((C_1 B_1)\) which is equivalent to Wiener measure, then (0.4) holds for \( f(F(x)) \). Thus the existence of a strong solution, \( \xi \), for the s.d.e. (0.1) is equivalent to the corresponding non-linear transformation, \( F \), being causally invertible.

Methods for determining when strong solutions exist are, in general, neither easy to develop nor apply. One such device is pathwise uniqueness as formulated by Yamada-Watanabe [3]. They showed that if two (weak) solution processes are pathwise identical, then the process is a strong solution. In Theorem 1, we present another condition which is both necessary and sufficient for existence of