THE HARMONIC MEASURE OF POROUS MEMBRANES IN $\mathbb{R}^3$

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In this paper several theorems concerning harmonic measure in Euclidean three space $\mathbb{R}^3$ are proved. We believe they are interesting in themselves, and in addition the extension of one of them, Theorem 2, to $\mathbb{R}^n$, $n \geq 4$, would permit an extension of the results of the authors' paper [2] to these dimensions.

Unless otherwise mentioned we will be working in $\mathbb{R}^3$, and a point $\bar{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ for which $x_2 = x_3 = 0$ will often be written $x_1$.

If $r$ is real put $B(\bar{x}, r) = \{z : |\bar{z} - \bar{x}| \leq r\}$, and if $A$ is a set define $|A - \bar{x}| = \inf_{\bar{y} \in A} |\bar{x} - \bar{y}|$ and $B(A, r) = \{\bar{y} : |\bar{y} - A| \leq r\}$. The (Newtonian) capacity of a compact set $E$ will be signified by $\text{cap} E$, and if $K$ is closed $u_K(\bar{z})$ will stand for the harmonic measure of $K$ relative to the point $\bar{z}$ and the region $\mathbb{R}^3$, that is for compact $E$

$$u_E(\bar{z}) = 0 \text{ if } \text{cap} E = 0 \text{ and } u_E(\bar{z}) = \int_E \frac{1}{|\bar{z} - \bar{x}|} d\gamma(\bar{x}) \text{ if } \text{cap} E > 0,$$

where $\gamma$ is the capacitary measure of $E$, while in general

$$u_K(\bar{z}) = \sup_E u_E(\bar{z}),$$

the supremum being taken over compact sets $E$ contained in $K$. The following theorem will be proved.

**THEOREM 1.** Given $\epsilon, \delta > 0$ there is a $\rho > 0$ such that if $F$ is a closed subset of $\mathbb{R}^3$, if $B(F, \rho)$ is connected and has diameter at least 1, and if

$$\text{cap}(F \cap B(\bar{x}, \rho)) \geq \delta \text{ cap } B(\bar{x}, \rho) = \delta \rho$$

for all $\bar{x} \in F$, then

$$u_F(\bar{z}) \geq (1 - \epsilon) \ u_{B(F, \rho)}(\bar{z}), \ \bar{z} \in \mathbb{R}^3.$$

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Since $u_B(0,r) = r/|z|$ if $|z| \geq r$, it is clear that the diameter condition of Theorem 1 cannot be entirely removed. Also, if $A = \bigcup B(\lambda i, r)$, $n$ and $r > 0$ fixed, then $\lim_{\lambda \to \infty} u_A(0)/u_B(A,r)(0) = 1$ so that the connectedness condition on $F$ cannot be replaced with a condition solely involving the number of balls of radius $\rho$ needed to cover $F$. The analogue of Theorem 1 for $R^4$ does not hold, as the following example shows. Extending our notation for a minute in the natural way to $R^4$, fix $r > 0$ and let $\Gamma = \{ \bar{x} \in R^4: \frac{4}{i=2} x_i^2 \leq r^2 \}$. Then

$$u_\Gamma(\bar{x}) = r(\frac{4}{i=2} x_i^2)^{-\frac{1}{2}} \text{ if } \frac{4}{i=2} x_i^2 \geq r^2,$$

so that

$$u_\Gamma(\bar{x})/u_B(\Gamma,r)(\bar{x}) = \frac{1}{2} \text{ if } \frac{4}{i=2} x_i^2 \geq (2r)^2,$$

while it is not hard to see that $\text{cap}(B(\bar{x},r) \cap \Gamma) \geq k \text{ cap } B(\bar{x},r)$ for all $x \in \Gamma$, where $k > 0$ does not depend on $\bar{x}$.

The following theorem is a corollary of Theorem 1 which we believe to hold, essentially unchanged, in $R^n$, $n \geq 4$.

THEOREM 2. Given $\epsilon, \delta > 0$ there exists $\rho > 0$ such that if $F$ is a closed set in $R^3$ which satisfies $B(0,1) \cap F = \emptyset$ and has the properties that every (continuous) path connecting $0$ to $\infty$ meets $B(F,\rho)$, and that

$$\text{cap}(F \cap B(\bar{x},\rho)) \geq \delta \text{ cap } B(\bar{x},\rho), x \in F,$$

then $u_F(0) \geq 1 - \epsilon$.

This result follows immediately from Theorem 1, for the path condition implies $u_B(F,\rho)(0) = 1$.

Before proving Theorem 1 we give an example of a result of [2] that we cannot now extend to higher dimensions but could, if we could extend Theorem 2.

THEOREM 3. For each $\epsilon > 0$ there is a $C(\epsilon) < \infty$ such that if $\Omega$ is a domain in $R^3$ with Green function $g_\Omega$, and if $\bar{x}, \bar{y}$ are points in $\Omega$ such that $g_\Omega(\bar{x},\bar{y}) \geq \epsilon|\bar{x}-\bar{y}|$, then $\bar{x}$ can be connected to $\bar{y}$ by a path in $\Omega$ of length at most $C(\epsilon) |\bar{x}-\bar{y}|$.

An appropriate four dimensional version of this theorem would be implied by the statement that for each positive $\epsilon$ there exists $K(\epsilon)$ such that if $\bar{x}, \bar{y} \in D$, for some domain $D$ in $R^4$, if $|\bar{x}-\bar{y}| = 1$, 