Chapter 6

Tilting with additional structure: twosided tilting complexes

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6.1 Introduction

In this section we shall give Rickard's approach to derived equivalences made explicit by giving a complex of bimodules $X$ such that the left derived tensor product by $X$ is an equivalence of categories. We shall derive properties of the derived equivalence induced by this left derived tensor product and we shall derive properties which are true in general for any equivalence of derived categories. At the end we shall give a non trivial example of such a complex of bimodules in a very explicit form.

6.2 The theorem and its proof

The main theorem in this chapter is the following.

**Theorem 6.2.1** (Rickard [139]) Let $R$ be a commutative ring and let $A$ and $B$ be two $R$-projective $R$-algebras. If $A$ and $B$ are derived equivalent by a functor $F$, then the functor $F$ induces also a derived equivalence between $D^b(A \otimes_R A^{op})$ and $D^b(A \otimes_R B^{op})$. The image of $A$ as $A \otimes_R A^{op}$ module under this functor is a complex, say $X$, in $D^b(A \otimes_R B^{op})$. Then,

$$X \otimes^L_B - : D^b(B) \longrightarrow D^b(A)$$

is an equivalence of triangulated categories.

A generalization and a more conceptual proof of this theorem will be given in section 8.3.2 by Bernhard Keller.

**Definition 6.2.1** The complex $X$ as in Theorem 6.2.1 is called a *twosided tilting complex*. Note that $X$ by definition is an object in the derived category and not in the homotopy category.
The proof of theorem 6.2.1 occupies all of section 6.2. Our proof presented here follows Rickard's original proof in [139]. We begin the proof with the following proposition which establishes already parts of the theorem.

**Proposition 6.2.2** Let $R$ be a commutative ring and let $\Lambda_0$ and $\Lambda_1$ be two $R$-algebras.
Let $T$ be a tilting complex over $\Lambda_0$ with endomorphism ring $\Gamma_0$.
Let $S$ be a tilting complex over $\Lambda_1$ with endomorphism ring $\Gamma_1$.
Suppose that for each $i \geq 1$ one has $\text{Tor}_i^R(\Lambda_0, \Lambda_1) = 0 = \text{Tor}_1^R(\Gamma_0, \Gamma_1)$.
Then, $S \otimes_R T$ is a tilting complex over $\Lambda_0 \otimes_R \Lambda_1$ with endomorphism ring $\Gamma_0 \otimes_R \Gamma_1$.

For the proof of this proposition we have to prove a property of the right derived functor of the Hom functor. We introduced the right derived functor at an abstract level in section 2.7.

**Lemma 6.2.3** Let $A$ be a ring and let $(X, d_X)$ and $(Y, d_Y)$ be objects in $D^-(A)$. The complex $R\text{Hom}_A(X, Y)$ is formed by the following ingredients The $n$-th homogeneous component $R\text{Hom}(X, Y)^n$ of $R\text{Hom}(X, Y)$ is
\[
R\text{Hom}_A(X, Y)^n := \prod_{p \in \mathbb{Z}} (X^p, X^{n+p})
\]
and the differential is
\[
d : R\text{Hom}_A(X, Y)^n \rightarrow R\text{Hom}_A(X, Y)^{n+1}
\]
\[
d : f \mapsto d_Y \circ f - (-1)^n f \circ d_X
\]
Then, $H^k(R\text{Hom}(X, Y)) \simeq \text{Ext}^k_A(X, Y) := \text{Hom}_{D^-(A)}(X, Y[k]).$

Proof of lemma 6.2.3. First of all with this definition $(R\text{Hom}_A(X, Y), d)$ is a complex. In fact, let $f \in R\text{Hom}_A(X, Y)^n$.
\[
d(f) = d(d_Y \circ f - (-1)^n f \circ d_X)
\]
\[
= d(d_Y \circ f) - (-1)^n d(f \circ d_X)
\]
\[
= d_Y \circ d_Y \circ f + (-1)^n d_X \circ d_Y \circ f - (-1)^n d_Y \circ d_X \circ f + (-1)^n d_X \circ d_X \circ f
\]
\[
= 0
\]
since $d_X \circ d_Y = d_Y \circ d_X$. Moreover, setting $d^n$ the restriction of $d$ to the $n$-th homogeneous component, we see $\text{ker } d^n = \text{Hom}_A(X, Y[n])$, the complex morphisms of degree $n$ between $X$ and $Y$. The image of $d^{n-1}$ in $\text{ker } d^n$ are precisely the mappings homotopic to 0 in $\text{Hom}_A(X, Y[n])$. This proves the lemma.

Proof of proposition 6.2.2. Since both $\Lambda_0$ and $\Gamma_0$ are $R$-projective, the tensor product (over $R$) of a projective $\Lambda_0$-module with a projective $\Gamma_0$-module is a projective $\Lambda_0 \otimes_R \Gamma_0$-module, and the complex $T \otimes_R S := \text{tot}(T \otimes_R S)$ is a bounded complex of finitely generated projective $\Lambda_0 \otimes_R \Lambda_1$-modules.
We fix an $n \in \mathbb{N}$. We form the quadruple complex
\[
X := \text{Hom}_{\Lambda_0 \otimes_R \Lambda_1}(T \otimes_R S, T \otimes_R S).
\]