In this chapter, we will discuss a certain way of globalizing the formulas for fundamental classes of Schubert varieties in isotropic Grassmannians from Chapter 3. Let, for simplicity, $X$ denote a nonsingular (complex) variety. Suppose that $V$ is a vector bundle on $X$, endowed with an everywhere nondegenerate bilinear form $\phi$. We will consider three cases:

$(C_n)$ Lagrangian: $\text{rank}(V) = 2n$ and $\phi$ is skew-symmetric;

$(B_n)$ odd orthogonal: $\text{rank}(V) = 2n + 1$ and $\phi$ is symmetric;

$(D_n)$ even orthogonal: $\text{rank}(V) = 2n$ and $\phi$ is symmetric.

Recall from the previous chapter the question raised by J. Harris (originally in the even orthogonal case):

Suppose that $E$ and $F$ are maximal isotropic subbundles of $V$. Find a polynomial in the Chern classes of $E$ and $F$, which represents the locus

$$\{ x \in X : \dim(E \cap F)(x) \geq k \},$$

where $k$ is a fixed positive integer.

In this chapter, we will give a solution to this problem, considering, as in Chapter 6, a more general situation. Assume we have an isotropic flag

$$F_* : 0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_n = F$$

with $\text{rank}(F_i) = i$.

For a given sequence $a_* : 1 \leq a_1 < \ldots < a_k \leq n$ of integers, consider the closed subset $D(a_*)$ of $X$, defined by

$$D(a_*) := \{ x \in X : \dim(E \cap F_{a_p})(x) \geq p, \ p = 1, \ldots, k \}.$$

As we know from the previous chapter, in the even orthogonal case there exists two families of rank $n$ isotropic subbundles:

$$\{ E : \dim(E \cap F)(x) \equiv n \text{ (mod 2) } \forall x \in X \} \text{ and}$$

$$\{ E : \dim(E \cap F)(x) \equiv n + 1 \text{ (mod 2) } \forall x \in X \}.$$

Thus, if $a_k = n$, to get a nonempty locus, we must impose in the definition of $D(a_*)$ the additional assumption that $\dim(E \cap F)(x) \equiv k \text{ (mod 2)}$. 
A model for the $D(a_\ast)$'s is provided by Schubert varieties in isotropic Grassmann bundles. We have three types of isotropic Grassmann bundles corresponding to the Lagrangian, odd orthogonal and even orthogonal cases respectively:

$$\pi : LG_nV \to X, \quad \pi : OG_nV \to X \quad \text{and} \quad \pi : OG_n''V \to X$$

where $OG_nV$ (resp. $OG_n''V$) parametrizes isotropic rank $n$ bundles $E$ such that $\dim(E \cap F)(x) \equiv n \pmod{2}$ (resp. $\dim(E \cap F)(x) \equiv n + 1 \pmod{2}$). Denoting by $S$ the tautological rank $n$ subbundle on these Grassmann bundles, we have Schubert bundles which for $\mathcal{G} = LG_nV$ or $\mathcal{G} = OG_nV$ are defined by (we omit writing the pull-back indices)

$$\Omega(a_\ast) := \Omega(a_\ast, F_\ast) = \{ g \in \mathcal{G} : \dim(S \cap F_a)(g) \geq i \quad \forall i \}.$$ 

The same definition, with the additional condition $k \equiv n \pmod{2}$, gives Schubert bundles in $\mathcal{G}' = OG_n'V$: by imposing $k \equiv n + 1 \pmod{2}$, we get Schubert bundles in $\mathcal{G}'' = OG_n''V$. By a well-known universality property of Grassmannians there exists a morphism $s : X \to \mathcal{G}$ in the first two cases and $s = (s', s'') : X \to \mathcal{G}' \cup \mathcal{G}''$ in the third, such that $s^*S = E$. We have set-theoretically

$$D(a_\ast) = s^{-1}(\Omega(a_\ast)),$$

and using the canonical scheme structure on the Schubert bundles, we define a scheme structure on $D(a_\ast)$ by the scheme-theoretic inverse image.

The strategy used, in this chapter, to compute the fundamental classes of the $\Omega(a_\ast)$'s and $D(a_\ast)$'s is as follows. First, we compute $[\Omega(a_\ast)]$ using essentially two tools: a certain desingularization of $\Omega(a_\ast)$ and the class of the (relative) diagonal

$$\Delta := \text{Im}(\delta) \subset \mathcal{G} \times_X \mathcal{G},$$

where $\delta : \mathcal{G} \to \mathcal{G} \times_X \mathcal{G}$ is the diagonal embedding. (In the even orthogonal case, $\mathcal{G}$ denotes one of the components $\mathcal{G}'$ or $\mathcal{G}''$.)

The following desingularization of $\Omega(a_\ast)$ will be suitable to carry out the computations in the Lagrangian case (as a general rule, in this chapter, we will give a detailed account for the Lagrangian case, and just formulate analogs of the most important results for the orthogonal cases; a detailed treatment of the orthogonal cases can be found in [P-R5]).

Let $p : \mathcal{F} = \mathcal{F}(a_\ast) \to X$ be the flag bundle parametrizing the flags

$$A_1 \subset A_2 \subset \ldots \subset A_k \subset A_{k+1}$$

where $\text{rank}(A_i) = i$ and $A_i \subset F_a$ for $i = 1, \ldots, k$; $\text{rank}(A_{k+1}) = n$ and $A_{k+1}$ is Lagrangian. Let $D$ denotes the tautological Lagrangian rank $n$ vector bundle on $\mathcal{F}$. By universality of $\mathcal{G}$ there is a map $\alpha : \mathcal{F} \to \mathcal{G}$ such that $\alpha^*S = D$ (on points, $\alpha$ sends a flag from $\mathcal{F}$ to its last rank $n$ member). It is easy to check that $\alpha$ establishes an isomorphism over the open subset of $\Omega(a_\ast)$ parametrizing those $g$

\footnote{This strategy was worked out in [P4, §5], where we refer the reader for more details.}