THE OUTER REGULARIZATION OF FINITELY-ADDITIVE MEASURES
OVER NORMAL TOPOLOGICAL SPACES

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Contents. 1. Introduction. 2. Negligibly bordered sets. 3. Outer regularity and the Identity Theorem. 4. The outer regularization of a non-negative finitely additive measures. 5. The Lévy topology. 6. Remarks on uniform tightness. A. Appendix: Two Principles of Caratheodory.

1. Introduction

Let $\Omega$ be a compact, convex subset of a locally convex topological vector space $X$, and $\partial_e \Omega$ be the set of all extreme points of $\Omega$. Then the integral form of the Krein-Milman Theorem asserts that for all $w_0 \in \Omega$, there exists a countably-additive probability measure $\mu_{w_0}(\cdot)$ on the $\sigma$-algebra $\mathcal{B}_{\Omega}$ of Borel subsets of $\Omega$ such that

\begin{equation}
\mu_{w_0}(\partial_{e} \Omega) = 1 \quad \text{and} \quad w_0 = \int_{\partial_{e} \Omega} \mu_{w_0}(dw),
\end{equation}

the last being a Pettis integral with values in $X$, and $\partial_{e} \Omega$ being the closure of $\partial_e \Omega$. This result is stated and proved in Phelp's book [6]; see the second italicized statement on p.6.

This theorem and its improvements and generalizations by Choquet and his followers provide the simplest and most direct method of proving any integral representation theorem of analysis or probability for which we have a prior determination of the underlying compact convex set $\Omega$ and of the crucial subsidiary set $\partial_e \Omega$. Indeed, Phelps proves several integral theorems in this way, e.g. the one of Bernstein on completely monotone functions, cf. [6, pp.11-16]. In recent years Professor Erik Thomas has very considerably enlarged the range of applicability of this approach by considering extreme generators of cones, instead of merely extreme points of compact convex sets, and by bringing to bear the concepts of conical measure and of the reproducing kernel of a Hilbert subspace of $X$, cf. e.g. [8].

Now we know what the set $\partial_{e} \Omega$ is, for the closed unit ball $\Omega$ of the dual space $X$ (under the weak* topology) of the Banach space $C(S)$ of continuous functions on a compact Hausdorff space $S$, cf. [2, p.441,
Hence the Riesz Theorem, which gives an integral representation for any vector in $X$, is very easily deducible from (1.1). But a spurious obstacle prevents us from establishing the Riesz Theorem in this very straightforward way: Phelps uses up the Riesz Theorem in his proof of (1.1)! Specifically, he uses the Riesz Theorem fully to show the compactness, under a suitable topology, of the space $\mathbb{M}$ of all subprobability measures on $\mathcal{A}_\sigma$—a result which seems to be crucial to any proof of (1.1), cf. [6, p.6].

We have felt for some time that the placement of the Riesz Theorem ahead of the Krein-Milman in the mathematical edifice is an architectural blemish, cf. [4, pp.428-429]. The present paper is an outcome of efforts to remove this blemish. We define directly the relevant topology $\ell$ for the space $\mathbb{M}$ of all non-negative real-valued measures on $\mathcal{A}_\sigma$, and show directly that the subspace $\mathbb{M}_1$ of subprobability measures on $\mathcal{A}_\sigma$ is compact in the $\ell$-topology. In this way we free the Phelps proof of (1.1) from its dependence on the Riesz Theorem, and thereby make possible the revelation of the latter as an exemplification of the former. This architectural undertaking has yielded some unforeseen new results and insights, which have an interest of their own.

Phelps can introduce his topology for the space $\mathbb{M}$ of measures by duality from the weak* topology, cf. [4, p.6]. We, on the other hand, having denied ourselves access to duality, have to define this topology $\ell$ directly in measure-theoretic terms. This is done in §5 in the general setting of a normal Hausdorff space $\Omega$, and for the space $\mathbb{M}$ of all non-negative, finitely additive, outer-regular measures on any (finitely additive) algebra $\mathfrak{A}$ containing the open subsets of $\Omega$. The normality of $\Omega$ and the outer-regularity of the measures in $\mathbb{M}$ are needed to secure the Hausdorff property of the topology $\ell$, cf. Lma. 5.5.

We use a direct, duality-free transfinite argument to establish our main theorems 5.15 and 5.16 on the $\ell$-compactness of the subspace $\mathbb{M}_1$ of subprobability measures in $\mathbb{M}$ (i.e. $\mu$ in $\mathbb{M}$ for which $\mu(\Omega) \leq 1$). This argument rests on combining Tychonov's Theorem with a new preliminary theorem on the outer-regularization of an (irregular) finitely additive measure on the algebra $\mathfrak{A}_0$ generated by the open subsets of $\Omega$. The study of outer-regularization is made in §4 and culminates in two main theorems 4.8 and 4.9. The proofs of the latter theorems are based on an adaptation of the beautiful Caratheodory method, cf. Appendix, which has proved useful in so many situations in abstract and topological measure theory. Our adaptation is in many ways similar to the one made by Halmos to extend a content over a locally compact Hausdorff space to a measure, cf. [3, §§53, 54].