AFFINE LIE ALGEBRAS AND MODULAR FORMS

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Introduction

This lecture is in some sense a sequel to that of Demazure [4], although its point of view will be somewhat different.

Affine Lie algebras are particular examples of Lie algebras defined by Cartan matrices, or Kac-Moody Lie algebras. These are infinite-dimensional complex Lie algebras defined by generators and relations, for which there exists a satisfactory structure theory and representation theory which mirrors precisely (and includes) the classical theory of finite-dimensional complex semisimple Lie algebras, and culminates in an analogue of Weyl's character formula and denominator formula. In the case of affine Lie algebras, these formulas can be made quite explicit, at any rate for certain modules, and lead to formal identities for theta-functions and modular forms. The simplest example is that of the trivial representation, which leads to the so-called denominator formula; this is an identity between formal power series in several variables, and can be specialised to give a large number of identities for Dedekind's \( \eta \)-function.

Apart from these connections with arithmetic and modular forms, which form the subject of this lecture, it has become apparent in the last few years that affine Lie algebras have connections with many other areas of mathematics: combinatorics (partitions, Rogers-Ramanujan identities) [5, 21]; topology (loop spaces and loop groups) [8, 9, 20]; linear algebra (representations of quivers) [14]; singularities [26]; completely integrable systems [1, 2] and the structures of mechanics and particle physics [6, 7]. There appear also to be tantalising but as yet little
understood connections with the "Monster" simple group [3,15]. The range of these applications, all of which are in a stage of active development, continues to increase at an alarming rate.

1. Finite-dimensional simple Lie algebras

In order to set the scene, we shall briefly review some of the salient facts about a finite-dimensional complex simple Lie algebra \( g \). Let \( h \) be a Cartan subalgebra of \( g \) (i.e., a maximal abelian diagonalizable subalgebra); let \( h^* \) be the dual of \( h \), and \( l \) the dimension of \( h \). There exists a non-degenerate symmetric bilinear form \((x, y)\) on \( g \) which is invariant, i.e.,

\[
([x, z], y) = (x, [z, y])
\]

for all \( x, y, z \in g \), for example the Killing form \( \text{tr}(\text{ad}(x)\text{ad}(y)) \). The restriction of this form to \( h \) is non-degenerate and hence determines a symmetric bilinear form \((\lambda, \mu)\) on \( h^* \).

Root system. For each \( \alpha \in h^* \) let \( g_\alpha \) denote the set of \( x \in g \) such that

\[
[h, x] = \alpha(h)x \quad \text{for all} \quad h \in h.
\]

Then \( g_0 = h \), and the non-zero \( \alpha \in h^* \) such that \( g_\alpha \neq 0 \) are the roots of \( g \) relative to \( h \). They form a finite subset \( R \) of \( h^* \), called the root system of \( (g, h) \). We have

\[
g = h^+ \sum_{\alpha \in R} g_\alpha
\]

and each \( g_\alpha \) is 1-dimensional. For each \( \alpha \in R \), the only roots proportional to \( \alpha \) are \( \pm \alpha \). The bilinear form on \( g \) may be chosen so that \( |\alpha|^2 = (\alpha, \alpha) \) is real and positive for each \( \alpha \in R \).

It is possible to choose roots \( \alpha_1, \ldots, \alpha_l \in R \) such that each root \( \alpha \in R \) is of the form \( \alpha = \sum n_\alpha \alpha_i \) with integer coefficients \( n_\alpha \), which are either all \( \geq 0 \) (positive roots) or all \( \leq 0 \) (negative roots). The \( \alpha_i \) are called a set of simple roots or a basis of \( R \), and we shall assume that they have been chosen once and for all. There is then a unique highest root, for which \( \sum n_\alpha \) is a maximum.

Weyl group. For each \( \alpha \in R \), let \( w_\alpha \) denote the reflection in the hyperplane orthogonal to \( \alpha \) in \( h^* \). We have