Chapter 11

Saint–Venant’s problem

Most of the theory developed above originated from the questions related to Saint–Venant’s problem which will be treated now. It is concerned with the connection between the three-dimensional beam theory and the one-dimensional rod theory.

It is proved in [Mi88c] that all equilibrium deformation of an infinite beam without external forces and having only small bounded strains are actually contained in a finite-dimensional center manifold. Moreover, the invariance of the problem under rigid-body transformations was exploited to show that the flow on the center manifold is described by a differential equation having exactly the form of the classical rod equations of Kirchhoff [Ki59] and Antman [An72]. This reduction process gives a mathematically rigorous way for deriving the corresponding (nonlinear) material law of the rod from the material law of the material constituting the three-dimensional beam.

The question, left unanswered in previous work, is whether starting with a hyperelastic three-dimensional material leads necessarily to a hyperelastic rod. Hyperelasticity just means that the equilibrium equations are found as the Euler–Lagrange equations with respect to some deformation energy density, usually called the stored-energy function. In the context of the present work this question is exactly the one of reducing a Lagrangian system onto its center manifold.

We derive the associated reduced stored-energy functional for the rod up to terms of third order. On this way we encounter the so-called Saint–Venant solutions of linearized elasticity which play a basic role in constructing the center space for the beam problem. Thus, we are able to avoid the deficiencies of classical projection methods which were not able to find correct values for the torsional rigidity (cf. the discussion at the end of Section 11.3). Moreover, we are able to show that, up to this order, the system allows for natural reduction. This provides the satisfactory result that the projection method, when applied with care, gives the correct result.

The resulting rod equations will be studied briefly in Section 11.5, for a complete discussion we refer to [HM88]. We mainly emphasize how the classical reduction via
symmetry can be applied to the reduced rod model obtained via the center manifold approach. In the general case the reduced phase space (see Section 5.1) is four-dimensional, and contains chaotic dynamics.

Cross-sectional symmetries and reflection symmetry with respect to a cross-section (reversibility) are studied in the last section.

11.1 The physical equations

Let \( \Omega = \Omega_l = \Sigma \times (-l, l) \) be the undeformed prismatic body, where the cross-section \( \Sigma \) with coordinates \( y = (y_1, y_2) \) is a bounded region in \( \mathbb{R}^2 \) with \( C^2 \)-boundary \( \partial \Sigma \) and where \((-l, l)\) is the domain of the axial coordinate \( t \). For a deformation \( \varphi : \Omega_l \to \mathbb{R}^3 \) the symmetric strain tensor \( E = E(\nabla \varphi) \) is given by \( \frac{1}{2}(\nabla_x \varphi \cdot \nabla_x \varphi - I) \), where \( x \) stands for \((y_1, y_2, t)\). For hyperelastic materials, the material behavior is characterized by the stored-energy function \( W = W(y, t, \nabla_x \varphi) \). By the invariance of the material properties under rigid-body transformations (also called frame indifference or objectivity) we know that \( W \) has to satisfy the relation \( W(x, RF) = W(x, F) \) for every orthogonal matrix \( R \). This implies that \( W \) is actually only a function of \( E \), viz. \( W(x, F) = W(x, E) \). For simplicity, we assume that the material is homogeneous in the axial direction, i.e. \( W \) does not depend on \( t \), and that the expansion

\[
\tilde{W}(y, E) = \frac{\lambda}{2}(\text{tr } E)^2 + \mu \text{tr}(E^2) + \mathcal{O}(|E|^3) \quad \text{for } E \to 0, \tag{11.1}
\]

holds, where the Lamé constants \( \lambda \) and \( \mu \) are positive and \( y \)-independent. By rescaling we may assume \( \mu = 1 \) to simplify subsequent formulae.

The equilibrium equations for a beam loaded only at its terminal surfaces \( \Sigma \times \{\pm l\} \) are the Euler-Lagrange equations for the functional

\[
I(\varphi) = \int_{\Omega_l} W(y, \nabla_x \varphi) \, dx + \int_{\Sigma} (\varphi(l, \cdot) g_+ - \varphi(-l, \cdot) g_-) \, dy.
\]

The Cauchy stress tensor \( T(y, \nabla_x \varphi) \) is given by \( \frac{\partial}{\partial y} W \), i.e. \( T_{ij} = \frac{\partial}{\partial y_i} W \) and has the form \( T(y, F) = FS(y, E) \), where the symmetric Piola-Kirchhoff tensor \( S \) is given by \( \frac{\partial}{\partial E} \tilde{W} \). The Euler-Lagrange equations are

\[
\text{div}(T(y, \nabla_x \varphi)) = 0 \quad \text{in } \Omega_l, \tag{11.2}
\]
\[
T(y, \nabla_x \varphi)n = 0 \quad \text{on } \partial \Sigma \times (-l, l), \tag{11.3}
\]
\[
T(y, \nabla_x \varphi)n = g_{\pm} \quad \text{for } (y, t) \in \Sigma \times \{\pm\}. \tag{11.4}
\]

Equation (11.2) is strongly elliptic for small \( E \) due to our assumption \( \lambda, \mu > 0 \). In fact, strong ellipticity is a reasonable global assumptions on \( W \) (cf. [Ba77]).