A nonempty set \( E \) is called a vector space over a field \( K \) if

(a) \( E \) is an additive abelian group, and
(b) for every \( a \in K \) and \( x \in E \), there is defined an element \( ax \) in \( E \) subject to the following conditions:

\[
\begin{align*}
(b_1) \quad & ax + ay = ax + ay \\
(b_2) \quad & (a + b)x = ax + bx \\
(b_3) \quad & a(bx) = (ab)x \\
(b_4) \quad & 1x = x
\end{align*}
\]

and (a, \( \beta \in K \), \( x, y \in E \) and \( 1 \) the unit element of \( K \) under multiplication.

If \( K \) is the field \( \mathbb{R} (\mathbb{C}) \) of real (complex) numbers, the vector space \( E \) is called a real (complex) vector space.

Throughout this book, we deal with only real or complex vector spaces, and we use \( 0 \) to denote the zero element of \( K \) as well as that of a vector space.

**Proposition 1.** If \( E \) is a vector space over \( K \),

(a) \( aO = O \) for all \( a \in K \);  
(b) \( 0x = 0 \) for \( x \in E \);  
(c) \( (-a)x = -(ax) \) for \( a \in K, x \in E \);  
and (d) \( ax = 0, x \neq 0, \) implies that \( a = 0 \).

A vector space \( E \) with a multiplication (that is, if \( x, y \in E \), then \( xy, yx \in E \)) is called an algebra.
If $E$ is a vector space and $F$ a nonempty subset of $E$, then $F$ is called a vector subspace (or simply, subspace) of $E$ if, under the operations of $E$, $F$ itself forms a vector space over the field $K$. If $x_1, \ldots, x_n \in E$, then $\sum_{i=1}^{n} \alpha_i \cdot x_i$, $\alpha_i \in K$, is called a linear combination of $x_1, \ldots, x_n$. A subset $B$ of a vector space $E$ is called linearly independent if $B \neq \emptyset$ or $\{0\}$ and no element of $B$ is a linear combination of any finite subset of other elements of $B$. A maximal linearly independent subset of a vector space is called a Hamel basis (or vector basis). Every vector space has a Hamel basis and any two Hamel bases of a vector space have the same cardinal number. The cardinal number of a Hamel basis of a vector space is called its dimension.

If $F$ is a subspace of a vector space $E$ over the field $K$, the quotient space of $E$ by $F$ is a vector space $E/F$ over $K$ where, for $x_1 + F, x_2 + F \in E/F$ and $\alpha \in K$,

(i) $(x_1 + F) + (x_2 + F) = (x_1 + x_2) + F$

and (ii) $\alpha(x_1 + F) = \alpha x_1 + F$.

An arbitrary product $E = \prod_{\alpha} E_{\alpha}$ of vector spaces $E_{\alpha}$ is a vector space where addition and scalar multiplication are defined as coordinatewise addition and scalar multiplication.

If $\{E_{\alpha}\}$ is a family of vector spaces and $F = \sum_{\alpha \in I} E_{\alpha} = \{x = \{x_\alpha\} ; \ x_\alpha = 0 \text{ for all } \alpha \text{ except for a finite subset of } I\}$, then $F$ is a vector space, called direct sum of $\{E_{\alpha}\}$, where addition and multiplication