CHAPTER I

TOPOLOGICAL VECTOR SPACES

Introduction

A topological space $E$ which is also a vector space over the field $K$ of reals or complexes is called a topological vector space if

(TVS1) the map $(x, y) \to x + y$ from $E \times E$ into $E$ is continuous, and

(TVS2) the map $(\lambda, x) \to \lambda \cdot x$ from $K \times E$ into $E$ is continuous.

A topology on a vector space is said to be compatible if the axioms (TVS1) and (TVS2) are satisfied.

REMARK 1. We shall usually deal with the Hausdorff topological vector spaces in the sequel.

THEOREM 1. Let $E$ be a topological vector space.

(a) For each $x \in E$ and $\lambda \in K$, $\lambda \neq 0$, the map $x \to \lambda \cdot x + x$ is a homeomorphism of $E$ onto itself.

(b) For any subset $A$ of $E$ and any basis $\mathcal{B}$ of the neighbourhood filter at $0$,

$$\bar{A} = \cap \{ A + V ; V \in \mathcal{B} \}$$

where $\bar{A}$ is the closure of $A$.

(c) If $A$ is an open subset of $E$ and $B$ any subset of $E$, then $A + B$ is an open subset of $E$.

(d) If $A$ is a closed subset and $B$ a compact subset of $E$, then $A + B$ is a closed subset of $E$. 
(e) If $A$ is a circled subset of $E$, so is its closure $\overline{A}$.

**THEOREM 2.** Let $E$ be a topological vector space. Then there exists a neighbourhood basis $\mathcal{B}$ of $0$ in $E$ such that

1. $(N_1)$ each $U$ in $\mathcal{B}$ is closed, circled and absorbing, and
2. $(N_2)$ for each $U$ in $\mathcal{B}$, there is a $V$ in $\mathcal{B}$ with $V + V \subseteq U$.

Conversely, if $E$ is a vector space and $\mathcal{B}$ is a filter basis satisfying $(N_1)$ and $(N_2)$, then there is a unique topology $\tau$ on $E$ which makes it a topological vector space and $\mathcal{B}$ is a neighbourhood basis at $0$.

A topological vector space $(E, \tau)$ is metrizable if there is a metric on $E$ whose open balls form a basis. A topological vector space $E$ is metrizable iff there is a countable neighbourhood basis at $0$. These neighbourhoods can be so chosen as to satisfy $(N_1)$ and $(N_2)$ of Theorem 2.

A subset $B$ of a topological vector space $E$ is called

1. (i) bounded if it is absorbed by every neighbourhood of $0$ in $E$, (ii) totally bounded (precompact) if for each neighbourhood $V$ of $0$ in $E$, there is a finite subset $B_0$ in $B$ such that $B \subseteq B_0 + V$.

Every totally bounded subset of a topological vector space $E$ is bounded.

A topological vector space $E$ is said to be

1. (i) complete if every Cauchy filter is convergent, (ii) quasi-complete if every closed and bounded subset of $E$ is complete, and (iii) sequentially complete (or, semi-complete) if every Cauchy sequence in $E$ converges.

Completeness $\Rightarrow$ quasi-completeness $\Rightarrow$ sequential