Chapter 2

INDIVIDUAL ERGODIC THEOREMS IN $L_2$ OVER A VON NEUMANN ALGEBRA

2.1. Preliminaries

Recently, a remarkable progress has been made in the individual ergodic theory of positive contractions in von Neumann algebras. Many pointwise ergodic theorems have been extended to the operator algebra context. The study of such problematics is motivated by the theory of open (irreversible) quantum-dynamical systems. From the physical point of view, the most important are completely positive maps on C*- or W*-algebras (see for example [71]) but in the context of this chapter it seems to be more natural to consider a larger class of positive contractions. We shall discuss the asymptotic behaviour of kernels in $M$ or rather their extensions to contractions in $H = L_2(M, \phi)$. More exactly, we shall prove several individual ergodic theorems concerning these contractions. Like in the classical ergodic theory the maximal inequalities are always behind such results and they will be discussed in the next section. Let us notice that this chapter is one of the main parts of this book. It is closely related to Chapter 2 (Ergodic Theorems) in [50]. In the forthcoming sections we shall also refer to [50] (trying to keep some notation).

We shall now briefly discuss the context of this chapter. The next section is devoted to some maximal ergodic inequalities which are the main tools in this and the next chapters. In section 2.3 we prove individual ergodic theorems for one and several contractions in $L_2(M, \phi)$ generated by kernels in $M$. Section 2.4 is devoted to the ergodic theorems with continuous time (including a local theorem in the spirit of Wiener) for weak*-continuous semigroups of Schwarz maps on $M$ preserving the state $\phi$ and their extensions to the contractions in $H$. In the last section 2.5 we prove a random ergodic theorem for some contractions in $H$.

2.2. Maximal ergodic lemmas

Let us begin with the following fundamental theorem which is a
natural and easy generalization of an important result of M.S. Goldstein [36]. Such version (for several kernels) will be needed in Chapter 4.

2.2.1. THEOREM (Goldstein's maximal theorem for several kernels). Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be kernels in $M$, $(\varepsilon_k)$ sequence of positive numbers, $(y_k) \subset M^+$. Set

$$s_n^{(i)} = n^{-1} \sum_{k=0}^{n-1} \alpha_i^k, \quad \text{for } i = 1, 2, \ldots, r; \quad n = 1, 2, \ldots.$$  

Then there exists a projection $P \in M$ such that

$$\phi(1 - P) \leq 2 \sum_{n=1}^{\infty} \varepsilon_n^{-1} \phi(y_n)$$

and

$$\|ps_n^{(i)}(y_k)p\| \leq 2\varepsilon_k, \quad \text{for } n, k = 1, 2, \ldots; \quad i = 1, \ldots, r.$$  

Proof. This is the trivial extension of Goldstein's result [36]. The proof can be easily obtained (mutatis mutandis) from the proof for one kernel [36]. We shall refer to [50, p. 19-21]. Instead of $L$ defined on page 19, we consider

$$L = \{(y_{n,k,i}) : 1 \leq n, k \leq N; \quad 1 \leq i \leq r, \quad y_{n,k,i} \in M^+; \quad \text{and } \sum_{n,k,i} y_{n,k,i} \leq 1\}.$$  

Then we put

$$g_n^{(i)} = \sum_{k=1}^{N} k[s_k^{(i)}(x_n)y_{n,k,i}^\omega, \Omega) - (y_{n,k,i}^\omega, \Omega)]$$

with $x_n = \varepsilon_n^{-1}x_n$, for $i = 1, 2, \ldots, r$ and set

$$g_n = \sum_{i=1}^{r} g_n^{(i)}, \quad g(y) = \sum_{n=1}^{N} g_n(y).$$

Then it is enough to repeat the reasoning in [50], p. 19-20.

The above theorem implies the following

2.2.2. THEOREM (Maximal Ergodic Lemma). For $i = 1, 2, \ldots, r$, let $a_{i,0}$ be a kernel in $M$. Denote by $a_i$ the contraction in $M$ generated by $a_{i,0}$. Put