1. Introduction. The theory of (qualitative) approximation by holomorphic functions of one or several variables has been a very active research area for a long time, and consequently a great deal of reasonably complete results are known, mainly in the one variable case [7,8,9,16]. In the last twenty years attention has also been paid to approximation problems by functions annihilated by differential operators other than $\partial$ (see for example [1,6,12]). More recently some new results have been discovered concerning approximation by functions annihilated by a (fixed) power of the Cauchy-Riemann operator $\partial = \partial / \partial \bar{z}$ on the complex plane. We will recall some of these results in section 2, and in section 3 we will present an extension of one of them to $\mathbb{C}^n$.

2. One variable results. Fix a compact subset $X$ of $\mathbb{C}$, and let $H(X)$ be the algebra of all functions which are holomorphic in a neighborhood of $X$. For $0 \leq m \in \mathbb{Z}$, let $H(m,X)$ denote the $H(X)$-module consisting of (the restrictions to $X$ of) the functions $f$ satisfying $\partial^{m+1} f = 0$ in a neighborhood (depending on $f$) of $X$. Write

$$A(m,X) = \{ f \in C(X) | \partial^{m+1} f = 0 \text{ in } \mathbb{C} \},$$

so that we have the obvious inclusion $H(m,X) \subset A(m,X)$.

Problem (O'Farrell [10]): describe those $X$ for which $H(m,X)$ is uniformly dense in $A(m,X)$.

Of course for $m=0$ this is the classical uniform approximation problem by rational functions (see [16]). The first theorem we want to quote is interesting because it shows there is a substantial qualitative difference between the classical problem ($m=0$) and the higher order case ($m>0$). One should remark that nevertheless this result has a surprisingly easy proof.

Theorem (Trent-Wang [13]). If $X$ has no interior points, then $H(1,X)$ (and therefore $H(m,X)$, $m > 1$) is dense in $C(X)$.

Thus the problem is reduced to compact sets with non empty interior, and, in fact, to compacts which are the closure of their interior. It turns out that the existence of interior points makes the problem much harder, as it does in the holomorphic case. The best known result for the case with non empty interior is the following extension of Mergelyan theorem ($m=0$ in the statement below).
Theorem (Carmona [4]). Suppose that $\mathcal{C} \setminus X$ has only finitely many connected components. Then $H(m,X)$ is dense in $A(n,X)$, $0 \leq m \in \mathbb{Z}$.

It is worth mentioning that for $m > 0$ no compact $X$ is known for which $H(m,X)$ fails to be dense in $A(m,X)$.

The result of Trent-Wang stated above is sharp only for $m=1$. In fact one gets a smoother degree of approximation when $m > 1$.

Theorem [15]. Suppose that $X$ has no interior points and let $f \in C^\infty(X)$. Then $f$ is approximable in the $\text{Lip}(m-1,X)$-norm by functions in $H(m,X)$, $1 \leq m$.

In [15] more precise results are stated involving Sobolev type norms. An interesting fact to point out is that in proving them one is naturally led to apply the theory of Calderón-Zygmund operators of homogeneous type.

Further results concerning approximation by more general $H(X)$-modules can be found in [3] and [14].

3. The main result. We are going to discuss an extension to several variables of the particular case of Carmona's result in which $m=1$ and $X$ is the closure of a domain with smooth boundary.

Theorem. [2]. Let $D \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with a $C^3$ boundary. Then each function $f \in C(\bar{D})$ satisfying

\begin{equation}
\frac{\partial^2 f}{\partial \bar{z}_i \partial z_j} = 0, \quad 1 \leq i, j \leq n,
\end{equation}

in $D$, can be uniformly approximated on $\bar{D}$ by functions satisfying (1) in a neighborhood of $\bar{D}$.

Remarks. (a) If in the statement of the Theorem the system (1) is replaced by the Cauchy-Riemann system $\frac{\partial f}{\partial \bar{z}_i} = 0$, $1 \leq i \leq n$, then we obtain a well-known approximation theorem by holomorphic functions due to Henkin [8,p.663]. As we will see later, the fact that (1) is a system of second order equations makes our problem more difficult, so that new ideas and technical tools are required.

(b) As it is easily seen, a functions $f$ satisfies (1) in $D$ if and only if there are holomorphic functions in $D$, say $h_0, h_1, \ldots, h_n$, such that $f = h_0 + h_1 \bar{z}_1 + \ldots + h_n \bar{z}_n$ in $D$. Since $h_j = \frac{\partial f}{\partial \bar{z}_j}$, $1 \leq j \leq n$, the $h_j$ are determined by $f$. It may happen that $f$ extends continuously to $\bar{D}$ and even that the $h_j$ are unbounded functions. For example, take $D$ to be the unit ball, $h_j = 0$, $2 \leq j \leq n$, $h_1 = -h_0$ and $h_0(z) = (1-|z|^2)^{-\frac{1}{2}}$. Therefore the approximation problem we deal with cannot be trivially reduced to known approximation theorems by holomorphic functions.

(c). We do not know wether the theorem is true for domains with $C^2$ boundary. However, and example by Diederich and Fornaess [5] can be used to show that strict