On Equivariant Homotopy Theory

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Introduction

Many of the ideas of homotopy theory seem to belong most naturally to the category $\text{G-Top}_B$ of $G$-spaces over a given $G$-space $B$, where $G$ is a topological group. Both $\text{Top}_B$ (the case when $G$ is trivial) and $\text{G-Top}$ (the case when $B$ is trivial) have been extensively studied but $\text{G-Top}_B$ itself appears to have been somewhat neglected. In this note we discuss some aspects of the homotopy theory of $\text{G-Top}_B$ and show how certain fundamental results in $\text{G-Top}$ and $\text{Top}_B$ can be better understood from this point of view.

In the combined theory appropriate versions of the notions of absolute retract (AR) and absolute neighbourhood retract (ANR) play an important role. The definitions we adopt are based on Fox's characterization although one can, of course, make definitions more in the spirit of Borsuk and Čech. As we shall see in §2, the familiar properties of AR/ANR theory in $\text{Top}$ generalise easily to $\text{G-Top}_B$. In §3 we give some applications, including a version of the well-known theorem of Dold [4] on the fibre homotopy equivalences. The remaining sections are devoted to the proof of a theorem about $G$-AR's, where $G$ is a compact Lie group, which implies a recent result of our own on equivariant homotopy theory. Indeed it was the desire to place this result on a more satisfactory footing that led to the development of ideas in the present note.
2. AR/ANR Theory in $G$-$\text{Top}_B$

Let $B$ be a $G$-space, where $G$ is a topological group. We say that a $G$-space $E$ over $B$ is a $G$-AR over $B$ if it has the following extension property. Let $(Y,X)$ be a paracompact $G$-pair, i.e., a pair in which $Y$ is a paracompact $G$-space and $X$ is a closed invariant subspace. Let $\theta, \phi$ be $G$-maps such that the diagram shown below is commutative.

$$
\begin{array}{ccc}
X & \xrightarrow{\theta} & E \\
\downarrow{\psi} & & \downarrow{p} \\
Y & \xrightarrow{\phi} & B \\
\end{array}
$$

By an extension of $\theta$ over $\phi$, we mean a $G$-map $\psi: Y+E$ such that $\psi|X = \theta$ and $p\psi = \phi$. If such an extension always exists we say that $E$ is a $G$-AR over $B$.

Note that in making this definition it is not necessary to describe $X,Y$ as $G$-spaces over $B$ and $\theta, \phi, \psi$ as $G$-maps over $B$, since they acquire this character automatically from the structure of the diagram.

When $B$ is paracompact we see, taking $(Y,X) = (B,\emptyset)$, that if $E$ is a $G$-AR over $B$ then $E$ admits an equivariant section $s: B \rightarrow E$.

When $E$ is a $G$-AR over $B$ then $E$ is $G$-contractible over $B$: the composition $sp: E+E$ is $G$-homotopic to the identity over $B$.

We say that a $G$-space $E$ over $B$ is a $G$-ANR over $B$ if, given any paracompact $G$-pair $(Y,X)$ and $G$-maps $\theta, \phi$ over $B$ as before, there exists an open invariant neighbourhood $U$ of $X$ in $Y$ and