A REMARK ON THE CENTRAL LIMIT THEOREM IN BANACH SPACES

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It is known that in cotype 2 or type 2 Banach spaces, pregaussian bounded random variables satisfy the central limit theorem (CLT). Conversely, by an example of S.A. Chobanyan and V.I. Tarieladze [1], if in a Banach space $E$, every pregaussian bounded random variable satisfies the CLT, necessarily $E$ is of finite cotype; nothing more can be said about the cotype and type properties of $E$ since for example the spaces $l_p(B)$, where $B$ is a Banach space of cotype 2 and $1 \leq p < \infty$, behave in regard to the classical CLT like the usual $l_p$ spaces (cf. [5]). Our aim in this paper is to construct a Banach space of cotype $2 + \delta$ and type $2 - \delta$ for every $\delta > 0$ in which there exists a pregaussian bounded random variable which does not satisfy the CLT. The starting point to this note is the paper [2] by E. Giné and J. Zinn where an example in the same spirit is construct in $l_2(B)$ when $B$ is not of cotype 2.

Let $E$ be a real separable Banach space. If $X$ is an $E$-valued random variable, $(X_n)_{n \in \mathbb{N}}$ will denote a sequence of independent copies of $X$ and $S_n(X)$ the partial sum $X_1 + \ldots + X_n$. With these notations, $X$ is said to satisfy the central limit theorem (CLT) if the sequence $(S_n(X)/\sqrt{n})_{n \in \mathbb{N}}$ converges in law. It is known that $X$ is then necessarily pregaussian (i.e. $X$ is centered and there exists an $E$-valued Gaussian random variable with the same covariance structure as $X$) and $\lim_{t \to \infty} t^2 P(\|X\| > t) = 0$. 
In this paper we will be mainly concerned with Banach spaces $E$ of the form $l^p((B_k)_k \in \mathbb{N})$ where $1 \leq p < \infty$ and $(B_k)_k \in \mathbb{N}$ is a sequence of real separable Banach spaces; $l^p((B_k)_k \in \mathbb{N})$ denotes the Banach space consisting of all sequences $x = (x_k)_k \in \mathbb{N} \in \prod B^k$ with $\|x\| = \left( \sum_{k \in \mathbb{N}} \|x_k\|^p \right)^{1/p} < \infty$. When all the $B_k$ are identical, $l^p((B_k)_k \in \mathbb{N})$ will be denoted by $l_p(B)$ where $B = B_k$.

The following theorem proved by E. Gine and J. Zinn [2] (see also [7] as well as [5] for the case $1 \leq p < 2$) provided the first examples of Banach spaces of type 2 in which the CLT under the classical necessary conditions fails.

**Theorem.** Let $1 \leq p \leq 2$ and $B$ be a real separable Banach space. If $B$ is not of cotype 2, there exists in $l_p(B)$ a pregaussian random variable $X$ such that $\lim_{t \to \infty} t^{-p} \mathbb{P}(\|X\| > t) = 0$ which does not satisfy the CLT.

One should notice that the proof of this theorem is based on an additional necessary condition for the CLT in spaces of the form $l^p((B_k)_k \in \mathbb{N})$ which we state here as a lemma.

**Lemma.** Let $X = (X_k)_k \in \mathbb{N}$ be a random variable taking its values in $l^p((B_k)_k \in \mathbb{N})$ satisfying the CLT. Then:

$$\lim_{n \to \infty} \sum_{k \in \mathbb{N}} \mathbb{E}[\max_{1 \leq j \leq n} \frac{\|X_j\|}{\sqrt{n}}]^p \mathbb{I}[\|X_j\| \leq \sqrt{n}] = 0$$

**Proof.** We first assume that $X$ is symmetrically distributed. Then, by Lévy's and Hoffmann-Jørgensen's inequalities [3], for every $n$:

$$\sum_{k \in \mathbb{N}} \mathbb{E}[\max_{1 \leq j \leq n} \frac{\|X_j\|}{\sqrt{n}}]^p \mathbb{I}[\|X_j\| \leq \sqrt{n}]$$