MOMENT PROBLEMS IN HILBERT SPACE

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Introduction

In this work we present a completely general technique for obtaining a solution of the moment problem. By this we mean, given a complex-valued integrand \( h(t,\lambda) \), find necessary and sufficient conditions on a function \( T(t) \) in order that

\[
T(t) = \int_{-\infty}^{\infty} h(t,\lambda) \, d\mu(\lambda)
\]

for some non-negative bounded measure \( \mu(\lambda) \).

A virtue of our reproducing kernel approach is that the method extends with only slight modification to the Hilbert space setting. Thus \( T(t) \) may be an operator family and \( \mu(\lambda) \) a generalized spectral measure.

Definition. A \(*\)-parameter set \( S \) is a set \( S = \{r,s,t,\ldots\} \) along with an idempotent unary operation \( * \), \( r^{**} = r \), and at least one \(*\)-fixed element \( u \), \( u^* = u \).

Remark. Since any set may be endowed with the required operation \( * \) by setting \( r^* = r \) for all \( r \in S \), it is seen that \( S \) may be completely arbitrary. Nevertheless, in all our applications, \( S \) will have either algebraic or topologic structure.

Definitions. A reproducing kernel space \( (H,K) \) is an inner-product space of functions \( \phi(\cdot) : S \to H \) from a \(*\)-parameter set \( S \) into a complex Hilbert space \( H \) along with a kernel \( K(\cdot,\cdot) : S \times S \to L(H) \) defined on pairs of parameters into the linear (possibly unbounded) operators defined on \( H \). Further \( K(\cdot,r)x \in H \) for \( r \in S \) and \( x \in H \) and the reproducing property holds, i.e.,

\[
(\phi(\cdot), K(\cdot,r)x)_H = \langle \phi(r), x \rangle_H, \quad \phi \in H, \; x \in H, \; r \in S.
\]

In the event that \( H \) is taken as the space of complex numbers, \( H = \mathbb{C} \), the classical Aronszajn case, then \( K(r,s) \) is just a complex number for
Letting $K^*(r,s)$ denote the adjoint of $K(r,s)$ when it exists, we say $K$ is **Hermitian symmetric** if

$$K^*(r,s) = K(s,r), \quad r, s \in S.$$ 

We say $K$ **factors** if, for the $*$-fixed element $u$, 

$$K(r,s) = K(r,u)K(u,s), \quad r, s \in S.$$ 

Finally, $K$ is **positive definite** if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} <K(r_i, r_j)x_i, x_j> \geq 0$$

for all $n = 1, 2, \ldots$, $\{r_1, \ldots, r_n\} \subset S$, and $\{x_1, \ldots, x_n\} \subset H$.

**Definitions.** An operator $\tilde{T}$ in a Hilbert space $\tilde{H}$ is a dilation of the operator $T$ in the Hilbert space $H$, notation $T = \text{pr}\tilde{T}$, if $\tilde{H} \supset H$ and for $x \in \text{dom} \ T$,

$$\tilde{P}\tilde{T}x = Tx$$

where $\tilde{P}$ is the orthogonal projection of $\tilde{H}$ onto $H$.

**Remark.** If $T = \text{pr}\tilde{T}$, then on $H$, $T$, and $\tilde{T}$ have the same weak values, i.e.,

$$<Tx, y> = <\tilde{T}x, y>, \quad x \in \text{dom} \ T, \ y \in H.$$ 

**Definitions.** Let $T \subset L(H)^S$ be a collection of functions $T(\cdot): S \rightarrow L(H)$. A function $F(\cdot, \cdot, \cdot): Tx S^2 \rightarrow L(H)$ is a **linear spread** on $T$ if $F$ has the form

$$F(T(\cdot), r, s) = \sum_{p=1}^{N} c_p(r, s)T(g_p(r, s)), \quad r, s \in S$$

for some complex valued functions $c_p: S^2 \rightarrow \mathbb{C}$ and binary operations $g_p: S^2 \rightarrow S$, $p = 1, \ldots, N$.

In this event define $F^-$ to be

$$F^-(T(\cdot), r, s) = \sum_{p=1}^{N} \overline{c_p(r, s)}T(\overline{g_p(r, s)}), \quad r, s \in S.$$ 

**Definition.** A linear spread $F$ is **admissible for** $T(\cdot): S \rightarrow L(H)$ if the following equalities are satisfied, for $r, s, t, w \in S$,

(a1) $F(T(\cdot), r, u) = T(r),$

(b1) $F^-(F(T(\cdot), \cdot, t), r, s, t) = F(F(T(\cdot), \cdot, t)s, r),$

(b2) $F(F(V(\cdot), \cdot, t), r, s) = F(F(V(\cdot), r, \cdot), s, t),$

where $V(\cdot) = F(T(\cdot), w, \cdot),$$

and (a2) $F(V(\cdot), u, s) = V(s)$, where $V(\cdot) = T(\cdot)$ or $V(\cdot) = F(T(\cdot), r, \cdot)$,