1. INTRODUCTION

Let $X$ and $Y$ be topological spaces, $A \subseteq X$, and $f : A \rightarrow Y$ be a single-valued continuous mapping of the subspace $A$ into $Y$. By $CE(f)$ we denote the set of all subsets $Z$ of $X$ such that $A \subseteq Z \subseteq [A]$ and there exists a single-valued continuous mapping $f_Z : Z \rightarrow Y$ such that $f = f_Z | A$. In this paper we study the following questions:

1. Does there exist an element $A_m \in CE(f)$ such that $A_m = \bigcup_{Z \in CE(f)} Z$?

2. Does there exist a "useful" characterization of sets $A_m$ in terms of $B$-sets of a space $X$?

According to the Lavrentiev's Theorem ([3], p. 341), if $[A] = X$ and $Y$ is a completely metrizable space, then there exists a $G_6$-set $Z \in CE(f)$. In section 3 we strengthen Lavrentiev's result for more general spaces. In this case we apply the method of compactification and the Ponomarev's idea of extension of set-valued mappings [7].

Notation and terminology:

1. All spaces are assumed to be $T_1$-spaces.

2. By $\omega X$ we denote the Wallman's compactification of a space $X$.

3. By $C(X)$ we denote the set of all closed non-empty compact subsets of $X$.

4. A single-valued mapping $\theta : X \rightarrow C(Y)$ is called a set-valued or multi-valued mapping of $X$ in $Y$. The set $\Gamma(\theta) = \bigcup\{|x| \times \theta(x) | x \in X\} \subseteq X \times Y$ is called the graph of $\theta$. By $\mathcal{H}_X$ and $\mathcal{H}_Y$ we denote the natural projections of $X \times Y$ onto $X$ and $Y$ respectively. A mapping $\theta$ is said to be a usc-mapping if for every closed set $L \subseteq Y$ the set $\theta^{-1}L = \{x \in X | \theta(x) \cap L \neq \emptyset\}$ is closed in $X$. A cusc-mapping is a usc-mapping $\theta : X \rightarrow C(Y)$ such that the graph $\Gamma(\theta)$ is a closed set in the space $X \times Y$. Every usc-mapping of $X$ into the $T_2$-space $Y$ is a cusc-mapping [2].

5. $\tau$ is a fixed infinite cardinal, $\Omega(\tau)$ is the sequence of all ordinal numbers $\xi < \tau$. 


6. \[ \text{[.] or [.]}, \text{ is the closure operator in a space } X. \]
7. For standard notation consult \([1-4]\).
8. The cardinality of a set \( X \) is denoted by \( |X| \).

2. SET-VALUED MAPPINGS

We consider topological spaces \( X \) and \( Y \), a subset \( A \subset X \) and a set-valued mapping \( \theta : A \rightarrow C(Y) \). A pair \( (B, \psi) \) is called a continuous extension of \( \theta \), if it satisfies the following conditions:

1°. \( A \subset B \subset X \) and \( \Gamma(\theta) \subset \Gamma(\psi) \);
2°. \( \psi : B \rightarrow C(Y) \) is a cusc-mapping;
3°. If \( \phi : B \rightarrow C(Y) \) is a cusc-mapping and \( \Gamma(\phi) \subseteq \Gamma(\psi) \), then
\[
\Gamma(\phi) \subseteq \Gamma(\psi) .
\]

2.1. LEMMA. Let \( A \subset B \subset X \) and let \( \psi : B \rightarrow C(Y) \) be a continuous extension of mapping \( \theta : A \rightarrow C(Y) \), where \( |Y| > 2 \). Then \( B = \{A\} \) and \( \Gamma(\psi) = \Gamma(\theta) \). Moreover if \( \theta \) is a usc-mapping and \( Y \) is a \( T_1 \)-space, then
\[
\theta(x) = \psi(x) \quad \text{for every } x \in A .
\]

PROOF. The set \( \Gamma(\psi) \) is closed in \( B \times Y \). Therefore \( H = \Gamma(\theta) \subseteq \Gamma(\psi) \). We assume that there exists \( (a, b) \in \Gamma(\psi) \setminus H \). We fix some point \( c \in Y \setminus \{b\} \), an open set \( U \) of \( X \) and an open set \( V \) of \( Y \) such that \( a \in U \), \( \{b, c\} \cap V = \{b\} \) and \( (U \times V) \cap \Gamma(\theta) = \emptyset \). Therefore the set \( F = \Gamma(\psi) \setminus (U \times V) \) is closed in the space \( B \times Y \). The set \( F = P_X(F) \) is closed in \( B \) by Proposition 1.1 and the mapping \( \gamma : B \rightarrow C(Y) \), where \( \gamma(x) = P_Y((\{x\} \times Y) \cap F) \cup \{c\} \), is the cusc-mapping. By construction, we have \( \Gamma(\theta) \subseteq \Gamma(\psi) \) and \( \Gamma(\psi) \not\subset \Gamma(\phi) \). By the definition of continuous extension we have \( \Gamma(\psi) \subset \Gamma(\phi) \). This contradiction completes the proof.

2.2. COROLLARY. Let \( A \subset B \subset \Phi \subset X \) and let \( \phi : B \rightarrow C(Y) \), and \( \psi : B \rightarrow C(Y) \) be a continuous extension of a mapping \( \theta : A \rightarrow C(Y) \). Then
\[
\Gamma(\phi) = \Gamma(\psi) \cap (B \times Y).
\]

2.3. THEOREM. Let \( Y \) be a quasi-compact space and \( A \subset X \). Then for every mapping \( \theta : A \rightarrow C(Y) \) there exists a unique continuous extension \( \psi : B \rightarrow C(Y) \), where \( B = \{A\} \).

PROOF. Let \( F = \Gamma(\psi) = \Gamma(\theta) \) and \( \psi(x) = P_Y((\{x\} \times Y) \cap F) \) for every \( x \in B \). Then a mapping \( \psi : B \rightarrow C(Y) \) is a cusc-mapping and \( \Gamma(\theta) \subseteq \Gamma(\psi) = \Gamma(\theta) \). An appeal to Lemma 2.1 now completes the proof.

Let \( A \subset X \) and \( \theta : A \rightarrow C(Y) \) be a set-valued mapping. We say, that a mapping \( \theta \) is continuously \( M \)-extendible in a point \( x \in X \), if \( x \in \{A\} \), and there exists a continuous extension \( \theta_x : A \cup \{x\} \rightarrow C(Y) \) for a mapping \( \theta \) onto \( A \cup \{x\} \). If \( |\theta_x(x)| = 1 \) and \( x \in \{A\} \),