A Hamiltonian system of ordinary differential equations models the motion of a discrete mechanical system when no frictional forces are present. Such systems have the form:

\[
\begin{align*}
\dot{p} &= -H_q(p, q) \\
\dot{q} &= H_p(p, q)
\end{align*}
\]  

Here $H : \mathbb{R}^{2n} \to \mathbb{R}$, $p, q \in \mathbb{R}^n$, $\dot{p} = \frac{dp}{dt}$ etc. The system (1) may also be written as

\[
\dot{z} = JH_z(z)
\]

where $z = (p, q) \in \mathbb{R}^{2n}$ and $J = \begin{pmatrix} 0 & -\text{id} \\ \text{id} & 0 \end{pmatrix}$, id denoting the $n \times n$ identity matrix.

One of the important properties of (2) is that if $z(t)$ is a solution, then $H(z(t))$ is independent of $t$. Thus $H$ is an integral for the system (2) or said another way, "energy is conserved" and solutions of (2) lie on an energy surface $H = \text{constant}$. A major question of interest for (2) is what sort of conditions does one have to impose on $H$ so that the energy surface, say $H^{-1}(1)$, possesses a periodic solution of (2). During the past ten years, a lot of progress has been made on this question. Restricting ourselves to a fairly general setting, we will briefly survey this work and

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then describe some recent research of our own in this direction. Before doing
so, it is worth emphasizing now that although there have been several
approaches to this question, all of the recent progress involves applications
of the calculus of variations in some form or other.

Our survey begins with an older theorem due to Seifert [1] which is the
first global result in a general setting that we know of for (2).

**Theorem 1:** Suppose \( H(p,q) = \sum_{i,j=1}^{n} a_{ij}(q)p_ip_j + V(q) \) where

\[ V \in C^2(\mathbb{R}^n, \mathbb{R}) \quad \text{and} \quad D = \{ q \in \mathbb{R}^n \mid V(q) < 1 \} \]

is diffeomorphic to the unit ball in \( \mathbb{R}^n \) and \( V_q(q) \neq 0 \) on \( \partial D \)

and \( a_{ij} \in C^2(\mathbb{R}^n, \mathbb{R}) \) and the matrix \( (a_{ij}(q)) \) is uniformly
positive definite in \( D \).

Then there is a \( T > 0 \), points \( Q_1 \neq Q_2 \in \partial D \), and a solution \((p(t), q(t))\)
of (2) on \( H^{-1}(1) \) such that \((p(0), q(0)) = (Q_1, Q_2)\) and \((p(T), q(T)) =

\((0, Q_2)\).

Observing that \( H \) is even in \( p \), if we extend \( p(t) \) as an odd function
about 0 and \( T \) and \( q(t) \) as an even function about 0 and \( T \), the
resulting function is a \( 2T \) periodic solution of (2) on \( H^{-1}(1) \). Thus
solutions of the type found by Seifert are special periodic solutions of (2)
whose projection in \( \mathbb{R}^n \) bounces back and forth between two points on \( \partial D \).

Roughly speaking Seifert obtained these "bouncing orbits" as geodesics in a
Riemannian metric associated with the kinetic energy term \( \sum_{i,j=1}^{n} a_{ij}(q)p_ip_j \).

Seifert's work was generalized by A. Weinstein [2] who proved

**Theorem 2:** Suppose \( H(p,q) = K(p,q) + V(q) \) where \( V \) satisfies \((V_1)\) and

\( K \) satisfies

\[ K(0,q) = K_{\infty} < 0, \quad K \text{ is even and strictly convex in } p, \quad \text{and} \quad K(qp,q) \rightarrow -\infty \quad \text{as} \quad |q| \rightarrow \infty \]

uniformly for \( p \in S^{n-1} \) and \( q \in \partial D \).

Then the conclusions of Theorem 1 hold.