CHAIN RINGS AND VALUATIONS

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1. The subring $V$ of a (commutative) field $F$ is called a valuation ring of $F$ if for every $x \neq 0$ in $F$ either $x$ or $x^{-1}$ is in $V$. If $K$ is an extension field of $F$, $V$ a valuation ring of $F$, then there exists a valuation ring $V'$ of $K$ with $V' \cap F = V$, see [9], Chapter VI.

We like to consider the above described situation in the non-commutative case.

A subring $R$ of a division ring $D$ is called a valuation ring of $D$ if $x$ or $x^{-1}$ is in $R$ for every $0 \neq x$ in $D$. This is equivalent with the condition that $D$ is a skew field of quotients of $R$ and that for any $a, b$ in $R$, either $aR \subseteq bR$ or $bR \subseteq aR$ and similarly $Ra \subseteq Rb$ or $Rb \subseteq Ra$. If we allow zero divisors, then we call such a ring a chain ring and we call it right chain ring if only the condition on the right ideals is required. A ring is called invariant if all its one-sided ideals are two-sided. If $R$ is a valuation ring of $D$ then $R$ is invariant if and only if $R$ is invariant under all inner automorphisms of $D$. A ring is right invariant if all its right ideals are two-sided.

Chain rings, or more generally, right chain rings, do not only occur as valuation rings for skew fields ([4], [5], [7], [8]) but also as coordinate rings in geometry, ([6]) as localizations of rings with a distributive lattice of right ideals, ([1]) and as building blocks in structure theorems for, among others, non-commutative Dedekind rings and FPF rings.

We return to the above stated problem: given a valuation ring $R$ of a division ring $D$ and an extension $D'$ of $D$. Does there exist a valuation ring $R'$ of $D'$ with $R' \cap D = R$? Such a ring $R'$ is called an extension of $R$ in $D'$. We will show by an example that the answer is no in general, consider then the case where $D$ is the center of the finite dimensional division algebra $D'$ and finally the case where $D'$ is the skew field of quotients of an Ore extension of $D$.

2. We consider the following well known example. Let $D = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_i \in \mathbb{Q}\}$ be the skew field of quaternions over the rational numbers $\mathbb{Q}$ where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. This is a division ring of dimension four over its center $\mathbb{Q}$. The local rings $\mathbb{Z}_p = \{\frac{a}{b} \in \mathbb{Q} \mid p \mid b\}$ are valuation rings in $\mathbb{Q}$ for every prime $p$. We show that there do not exist valuation rings $B$ of $D$ with $B \cap \mathbb{Q} = \mathbb{Z}_p$ for $p \neq 2$. Assume that such a ring $B$ exists and denote by $M$ its maximal ideal. Then $B/M$ is a division ring finite dimensional over the field $\mathbb{Z}_p/M = GF(p)$ with $p$ elements, hence
finite and therefore commutative. However, \( i, j \) are elements in \( B \setminus M \) whose images do not commute in \( B/M \).

For \( p = 2 \) an extension \( B \) of \( \mathbb{Z}_2 \) in \( D \) can be defined as follows:

\[
B = \{ \alpha \in D \mid N(\alpha) \in \mathbb{Z}_2 \}
\]

where \( N(a_0 + a_1 i + a_2 j + a_3 k) = a_0^2 + a_1^2 + a_2^2 + a_3^2 \).

The property \( N(\alpha)N(\beta) = N(\alpha\beta) \) shows that \( B \) is closed under multiplication and also that \( B \) is a valuation ring of \( D \) provided \( N(\alpha + \beta) \) is in \( \mathbb{Z}_2 \) for \( \alpha, \beta \) in \( B \). This last property can be checked directly — of course, it is here where the fact \( p = 2 \) enters.

P.M. Cohn in [5] investigates, using completions, invariant extensions of valuation rings \( V \) in the center \( F \) of a finite dimensional division algebra \( D \). His main result is extended by Wadsworth in [8] as follows: Theorem. A valuation ring \( V \) of \( F \) has an invariant extension \( B \) in \( D \) if and only if \( V \) has a unique extension in every commutative subfield \( K \) of \( D \) with \( K \supseteq F \). It follows immediately from this result, that the number of invariant extensions of \( V \) in \( D \) is either zero or one.

3. We consider the following situation: Let \( D \) be a division ring with center \( F \) and \( [D : F] = n^2 \). Let \( V \) be a valuation ring of \( F \) and \( \mathcal{B} = \{ B \mid B \cap F = V \} \) be the set of valuation rings of \( D \) that intersect with \( F \) in \( V \). Then the following result holds ([3]):

**Theorem 1.** \( |\mathcal{B}| \leq n \) and any two extensions \( B_1 \) and \( B_2 \) of \( V \) in \( D \) are conjugate in \( D \).

To prove this theorem the following results are needed:

**Proposition 1.** Let \( |\mathcal{B}| \neq 0 \). Then there exists a valuation ring \( R \neq D \) of \( D \) such that \( B \subseteq R \) for all \( B \) in \( \mathcal{B} \).

In the next proposition let \( R \) be a valuation ring of \( D \) minimal with respect to the property of containing all extensions \( B \) of \( V \). Such a ring \( R \) is invariant and we denote with \( N \) its maximal ideal.

**Proposition 2.** Let \( |\mathcal{B}| > 1 \), \( Z \) the center of \( R/N \) and \( S \) the maximal separable extension of \( K_0 = R \cap N \) in \( Z \). Then, \( K_0 \) is a proper subfield of \( Z \), \( S \) is a Galois extension of \( K_0 \) and each \( K_0 \)-automorphism of \( S \) is induced by an inner automorphism of \( D \).

Theorem 1 is then proved using the above propositions, induction on \( n \) and Galois theory.

We saw earlier that \( \mathcal{B} \) can be empty. The next result shows that \( |\mathcal{B}| \geq 1 \) implies that the integral closure of \( V \) in \( D \) is a subring and can be described as the intersection of the extensions of \( V \) in \( D \) as it is the case in the classical commutative situation.