1. This note illustrates the usefulness of local time theory and the Cameron-Martin formula by giving two very short proofs of a stopped Brownian motion formula obtained in H.M. Taylor's paper [8] and applied there to problems in process control and in playing the stock market. The reader will easily find in the literature other applied problems for which local time theory may be effectively employed.

Let $\Omega$ be the space of continuous functions $\omega : [0, \infty) \rightarrow \mathbb{R}$. For $t \geq 0$, let $X_t$ be the $t$-th coordinate function so that $X_t(\omega) = \omega(t)$. Let $\mathcal{J}^0 = \sigma\{X_t : t \geq 0\}$ be the smallest $\sigma$-algebra on $\Omega$ which 'measures' each $X_t$.

Define

$$M_t = \max_{0 \leq s \leq t} X_s.$$

(As usual, we suppress the $\omega$'s.) Fix $a > 0$ and define

$$T = \inf\{t : M_t - X_t = a\},$$

so that $T$ is the first time that the process $X$ drops $a$ units below its maximum-to-date. We make the usual convention that $\inf\emptyset = \infty$. Note that $X_T = M_T - a$ on the set $\{T < \infty\}$.

On $(\Omega, \mathcal{J}^0)$ introduce the measures:

- $W$: Wiener measure, the law of standard Brownian motion starting at 0;
- $D_\mu$: the law of a Brownian motion starting at 0 with drift constant $\mu$ and variance coefficient 1, so that $D_0 = W$ and

$$\{X_t : t \geq 0 ; D_\mu\} \sim \{X_t + \mu t : t \geq 0, W\},$$

"~" denoting "is identical in law to".

If $P$ denotes any probability measure on $(\Omega, \mathcal{J}^0)$ and $\xi$ is (say) a positive random variable on $(\Omega, \mathcal{J}^0)$, we write

$$P[\xi] = \int \xi \, dP = \int \xi(\omega)P(\omega)$$

for the $P$-expectation of $\xi$. We use $E$ for expectation when the appropriate "$P$" is obvious.
Taylor's formula characterises the joint $D_{\mu}$-distribution of $X_T$ and $T$ as follows:

\[ D_{\mu} \{ \exp(\alpha X_T - \beta T) \} = \frac{\delta \exp[-(\alpha + \mu)a]}{\delta \cosh(\delta a) - (\alpha + \mu)\sinh(\delta a)} \]

which holds for $\alpha > 0$ and $\beta < \theta$, where

\[ \delta = [\mu^2 + 2\beta]^{\frac{1}{2}}, \quad \theta = \delta \coth(\delta a) - \mu > 0. \]

(Not\footnote{\textit{Note}. The discussion in Section 3 of Taylor's paper misses the fact that $M_T = X_T + a$ is exactly exponentially distributed. This follows either by an obvious "lack of memory" argument or directly from 1.1(i).})

Notice that because of the Cameron-Martin formula:

\[ D_{\mu} \{ \exp(\alpha X_T - \beta T) \} = \mathbb{W}[\exp(\alpha X_T - \beta T)\exp(\mu X_T - \frac{1}{2}T)] \]

(see §3.7 of McKean [5]), it is enough to establish the "$\mu = 0" \text{ case of (1.1) in the form:}

\[ \mathbb{W}[\exp(\alpha M_T - \beta T)] = \delta[\delta \cosh(\delta a) - \alpha \sinh(\delta a)]^{-1} \]

for $\beta > 0$ and $\alpha < \theta$, where, from now on,

\[ \delta = (2\beta)^{\frac{1}{2}}, \quad \theta = \delta \coth(\delta a). \]

For rigorous justification of (1.2), it is necessary to check that

\[ \mathbb{W}[T<\infty] = D_{\mu}[T<\infty] = 1. \]

The validity of (1.4) will become clear in a moment.

Lévy tells us that under $\mathbb{W}$, the process $Y = M - X$ is standard reflecting Brownian motion starting at 0 and $M$ is the local time $L(\cdot,0)$ at 0 for $Y$:

\[ M_t = L(t,0). \]

Recall that local time $L(t,y)$ at $y$ before $t$ for $Y$ may be defined as follows:

\[ L(t,y) = \lim_{\varepsilon \downarrow 0} (2\varepsilon)^{-1} \text{meas} \{ s \leq t : Y_s \in [y,y+\varepsilon) \}. \]

(We are not bothering about "almost surely", etc..) See McKean [5] for a recent expository article on Brownian local time which contains all of the results which we shall need.