The aim of the paper is to provide an answer to the following problem:

Let an algebraic torus $T$ act on a normal complete variety $X$. Describe all open $T$-invariant subsets $U$ of $X$ for which the geometric quotient space $U/T$ exists and is complete.

The problem has been studied in [B-B,S] for algebraic actions defined over the complex number field $\mathbb{C}$ and for complex analytic actions. Here we consider the algebraic case where the ground field $k$ is algebraically closed of any characteristic. The main result is in the spirit of [B-B,S], however our proof is based on completely different ideas. The proof gives in fact a result concerning more general quotients. The answer to the problem starting at the beginning follows directly from this result.

§1. Notations and Terminology. The main result.

The ground field $k$ is assumed to be algebraically closed; all algebraic varieties and their morphisms are supposed to be defined over $k$. Let $T$ denote a one-dimensional torus and let $X$ be a normal complete algebraic variety. Assume that we have an action of $T$ on $X$. For any $t \in T$ and $x \in X$, $tx$ denotes the value at $x$ of the automorphism of $X$ assigned to $t$.

Let $X = X_1 \cup \cdots \cup X_r$ be the decomposition of the fixed point set of the action given on $X$ into connected components.

Let us fix an isomorphism $T \cong k^*$. Then for any $x \in X$, the morphism $\phi_x : T \to X$, defined by $\phi_x(t) = tx$, can be extended to the projective line $\mathbb{P}^1(k) \to k^*$. The extended morphism will be also denoted by $\phi_x$. Define

$$\phi^+(x) = \phi_x(0), \quad \phi^-(x) = \phi_x(\infty)$$

$$X^+_i = \{x \in X_i \mid \phi^+(x) \in X_i \}, \quad X^-_i = \{x \in X_i \mid \phi^-(x) \in X_i \}.$$ 

Definition 1.1. Let $i, j \in \{1, 2, \ldots, r\}$. We say that $X_i$ is directly less than $X_j$ if there exists $x \in X - X^T$ such that $\phi^+(x) \in X_i$, $\phi^-(x) \in X_j$. We say that $X_i$ is less than $X_j$ and we write $X_i < X_j$ if there exists a sequence $i = i_0, \ldots, i_s = j$ such that $X_{i_s}$ is directly less than $X_{i_s}$, for $s = 1, \ldots, l$. We shall write $X_i \preceq X_j$ if $X_i < X_j$ or $X_i = X_j$.

Definition 1.2. A semi-section of $\{1, 2, \ldots, r\}$ is a division of $\{1, \ldots, r\}$ into three disjoint subsets $A^+, A^0, A^-$ satisfying the following condition:
if \( i \in A^+ \cup A^0 \) and \( X_j \subset X_i \) then \( j \in A^+ \).

A section of \( \{1,2,\ldots,r\} \) is a semi-section \( (A^+,A^0,A^-) \) where \( A^0 = \emptyset \).

**Definition 1.3.** Let \( (A^+,A^0,A^-) \) be a semi-section of \( \{1,2,\ldots,r\} \) and let
\[
U = \bigcup_{i \in A^+ \cup A^-} (X_i^+ \cap X_i^-)
\]
Then \( U \) is called a semi-sectional set corresponding to the semi-section \( (A^+,A^0,A^-) \). A semi-sectional set corresponding to a section is called a sectional set.

Notice, that if \( U \) is a semi-sectional set corresponding to a semi-section \( (A^+,A^0,A^-) \) then
\[
U = X - \left( \bigcup_{i \in A^+ \cup A^-} X_i^- \cup \bigcup_{i \in A^-} X_i^+ \right)
\]

In this paper we are going to consider two concepts of quotient maps: a geometric quotient and a semi-geometric quotient. The notion of a geometric quotient was introduced by Mumford in [G.I.T.]. In the special case, we are considering in this paper, his definition is equivalent to the following:

**Definition 1.4.** Let an algebraic torus \( T \) act on an algebraic variety \( X \). A morphism \( \pi: X \to Y \), where \( Y \) is an algebraic variety, is said to be a geometric quotient of \( X \) (with respect to the given action of \( T \)) if the following conditions are satisfied:

(a) for any \( y \in Y \), \( \pi^{-1}(y) \) is an orbit in \( X \),

(b) \( \pi \) is an affine morphism,

(c) for any open affine \( U \subset Y \), the ring \( k[U] \) of regular functions on \( U \) is identified by \( \pi^* \) with the ring \( k[\pi^{-1}(U)]^T \) of regular \( T \)-invariant functions on \( \pi^{-1}(U) \).

**Definition 1.5.** Let \( X, Y, \pi \) be as in Definition 1.4. The morphism \( \pi: X \to Y \) is said to be a semi-geometric quotient of \( X \) (with respect to the given action of \( T \)) if conditions (b) and (c) of Definition 1.4 are satisfied. (The notion of a semi-geometric quotient is equivalent to the notion of a good quotient of Seshadri, see [Se])

We are going to denote a semi-geometric quotient of \( X \) by \( X \to X/T \).

It is easy to see that if a semi-geometric quotient exists, then it is a categorical quotient. On the other hand if \( \pi: X \to Y \) is a categorical quotient and \( \pi \) is an affine morphism, then \( \pi: X \to Y \) is a semi-geometric quotient.

Now, we are ready to state the main result of the paper.

**Theorem.** Let \( X \) be a normal complete algebraic variety with an action of \( T \). If \( U \) is a semi-sectional subset of \( X \), then \( U \) is open, \( T \)-invariant, a semi-geometric