Abstract

We treat the quantum stochastic differential equation (*) \( U(t) = A(t)U(t) \) with \( A(t) = AF(t) - A^*F(t) \) where \( A \) is a complex \( d \times d \)-matrix and \( F(t) \) white quantum noise. Its Stratonovich solution yields the quantum analogue of a process with independent stationary increments on the unitary group \( U(t) \). Light emission and absorption may be modelled as a two level atom in contact with a heatbath of quantum oscillators and may be described by (*) with \( d = 2 \) and \( A = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} \). For a heatbath consisting of finitely many oscillators, \( F(t) \) is coloured quantum noise and (*) can easily be solved explicitly. If the heatbath gets infinite in an appropriate sense, \( F(t) \) approximates white quantum noise and the solution for the finite heatbath converges to the Stratonovich solution of (*) with white noise \( F(t) \).

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§ 0. Introduction

A stochastic differential equation as it stands is meaningless. One has to specify the method of solution. From a mathematical point of view Ito solutions are more convenient whereas from a physical point of view the Stratonovich procedure turns out to be more natural. In a previous paper [6] we discussed the Ito solution of a linear quantum stochastic differential equation which is connected with the study of light emission and absorption in the Wigner-Weisskopf-approximation. In this paper we treat the more physical Stratonovich solution of the same differential equation and discuss the associated physical problem. The main tool for the Stratonovich solution is a theorem on time ordered moments of white quantum noise, which has been proved in [4]. This theorem is restated in § 1 in a generalized form. The proof is the nearly same as in Ref. [4] and most of it will be slapped over from [4]. The Stratonovich solution itself is given in § 2. We now turn to the physics and indicate the context in which the mathematical problems occur.

In their simplest form, light emission and absorption can be described as an interaction of an atom with a radiation field. We consider a two-level atom interacting with a radiation field which is in thermal equilibrium, i.e. we treat the radiation field as a heatbath of oscillators. The rotating wave approximation is assumed throughout.

Let us start with a heatbath consisting of finitely many oscillators labelled by $\lambda$, $\lambda \in \Lambda$, with frequencies $\omega_0 + \omega_\lambda$ where $\omega_0$ is the transition frequency of the atom. Later on the transition to infinitely many oscillators will be performed. The oscillators are described by the usual creations and annihilation operators $B_\lambda, B^*_\lambda, \lambda \in \Lambda$, whose only non-trivial commutators are $[B_\lambda, B^*_\lambda] = 1$.

The two-level atom is described by the operators on $\mathbb{C}^2$, i.e. the complex 2 x 2-matrices in the Hilbertspace $\mathbb{C}^2$. We introduce the special matrices

\[
\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]

Together with the unit matrix they form a basis of $\mathbb{C}^{2 \times 2}$, the set of complex 2 x 2-matrices. The Hamiltonian of the atom is

\[
H_{\text{at}} = \omega_0 \sigma_3.
\]

The Hamiltonian of the radiation field is

\[
H_{\text{ph}} = \sum_{\lambda \in \Lambda} (\omega_0 + \omega_\lambda) B^*_\lambda B_\lambda
\]

and the atom-field interaction in rotating wave approximation is given