§ 10. QUADRATIC FORMS OVER FORMALLY REAL FIELDS

(Part II)

In this paragraph we continue the study of quadratic forms over formally real fields from § 2. We will introduce the Witt ring $W(F)$ of a field $F$ and study the connection between minimal prime ideals of $W(F)$ and orderings of $F$. Finally we present a characterization of the Witt rings corresponding to SAP-fields. Although there exists a large theory of Witt rings of fields (see [La]), we introduce only what we need to present the mentioned topics.

As in § 2 let $F$ be a field of characteristic $\neq 2$. First we state without proof (for a proof see [La]) Witt's Cancellation Theorem

(10.1) THEOREM - Let $\rho, \rho_1, \rho_2$ be quadratic forms over $F$. Then
\[
\rho \perp \rho_1 \perp \rho_2 \text{ implies } \rho \perp \rho_2.
\]

Recall that any quadratic form of dimension $n$ is equivalent to some diagonalization $<a_1, \ldots, a_n>$. Regularity holds iff all $a_i$ are $\neq 0$.

(10.2) PROPOSITION - A regular quadratic form $\rho$ is isotropic iff $\rho \propto <a, -a>$ for some $\rho'$ and $a \in F$.

Proof: For the non-trivial direction let $\rho$ be isotropic. Furthermore let $\rho \propto <a_1, \ldots, a_n>$. Then $0 = \sum_{i=1}^{n} a_i v_i^2$ for some $v_i \in F$, and without loss of generality $v_1 \neq 0$. Hence $-a_1 = \sum_{i=2}^{n} a_i (\frac{v_i}{v_1})^2$. Therefore the $(n-1)$-dimensional form $<a_2, \ldots, a_n>$ represents $-a_1$. By Corollary (2.5) this implies $<a_2, \ldots, a_n> \propto <-a_1, c_3, \ldots, c_n>$ for some $c_j \in F$. But then $<a_1, a_2, \ldots, a_n> \propto <a_1, -a_1, c_3, \ldots, c_n>$.

q.e.d.

(10.3) LEMMA - Let $\rho$ be a 2-dimensional regular quadratic form. Then (1) to (4) are equivalent:
(1) \( \rho \) is isotropic

(2) \( \rho \cong <a,-a> \) for some \( a \in \mathbb{F} \)

(3) \( \rho \cong <1,-1> \)

(4) \( d(\rho) = -1 \cdot \mathbb{F}^2 \).

Proof: (1) \( \iff \) (2) follows from (10.2), and (3) \( \Rightarrow \) (4) is trivial.

(4) \( \Rightarrow \) (2): Let \( \rho \cong <c,d> \). From (4) we get \( d(\rho) = cd \cdot \mathbb{F}^2 = -1 \cdot \mathbb{F}^2 \).

Hence \( d = -c \mod \mathbb{F}^2 \). Thus \( \rho \cong <c,-c> \).

(2) \( \Rightarrow \) (3): Since \( 1 = a(\frac{1+a}{2a})^2 - a(\frac{1-a}{2a})^2 \), by Corollary (2.5), \( \rho \cong <1,c> \). From \( d(\rho) = -a^2 \cdot \mathbb{F}^2 = c \cdot \mathbb{F}^2 \) we obtain \( c = -1 \mod \mathbb{F}^2 \).

Hence \( \rho \cong <1,-1> \).

q.e.d.

The 2-dimensional form \( <1,-1> \) is called the hyperbolic form or plane. A 2n-dimensional form \( \rho \) is called hyperbolic if \( \rho \cong n<1,-1> \).

(10.4) THEOREM (Witt [W]) - Every quadratic form \( \rho \) over \( \mathbb{F} \) admits a decomposition

\[
\rho \cong r<0> \perp s<1,-1> \perp \rho_a,
\]

where \( \rho_a \) is anisotropic. The natural numbers \( r \) and \( s \) and the equivalence class of \( \rho_a \) are uniquely determined by \( \rho \).

Proof: For the existence let \( \rho \cong r<0> \perp \rho_1 \) such that \( \rho_1 \) is regular. Now apply Proposition (10.2) till the desired form is reached. To prove

the uniqueness use Witt's Cancellation Theorem (10.1).

q.e.d.

\( \rho_a \) will be called the anisotropic part of \( \rho \).

We now introduce a new equivalence relation on the set of regular quadratic forms (of arbitrary dimension) over \( \mathbb{F} \). Let \( \rho_1 \) and \( \rho_2 \) be regular. Then \( \rho_1 \) is called similar to \( \rho_2 \) (we write \( \rho_1 \sim \rho_2 \)) if

\[
\rho_1 \perp n<1,-1> \cong \rho_2 \perp m<1,-1>.
\]