Stability of stochastic differential equations in manifolds

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Abstract. — We extend the so-called topology of semimartingales to continuous semimartingales with values in a manifold and with lifetime, and prove that if the manifold is endowed with a connection $\nabla$ then this topology and the topology of compact convergence in probability coincide on the set of continuous $\nabla$-martingales. For the topology of manifold-valued semimartingales, we give results on differentiation with respect to a parameter for second order, Stratonovich and Itô stochastic differential equations and identify the equation solved by the derivative processes. In particular, we prove that both Stratonovich and Itô equations differentiate like equations involving smooth paths (for the Itô equation the tangent bundles must be endowed with the complete lifts of the connections on the manifolds). As applications, we prove that differentiation and antidevelopment of $C^1$ families of semimartingales commute, and that a semimartingale with values in a tangent bundle is a martingale for the complete lift of a connection if and only if it is the derivative of a family of martingales in the manifold.

1. Introduction

Let $(\Omega, (\mathcal{F}_t)_{0 \leq t < \infty}, \mathbb{P})$ denote a filtered probability space, $M$ a smooth connected manifold endowed with a connection $\nabla$. Then the tangent bundle $TM$ inherits a connection $\nabla'$ (usually denoted by $\nabla^c$), the complete lift of $\nabla$ (see [Y-I] for details). Let $X$ be a continuous semimartingale with values in $M$. The antidevelopment of $X$
in $T_{X_0}M$ is the semimartingale $Z$ solving the Stratonovich equation

$$p(\delta Z) = U_0 U^{-1}\delta X, \quad Z_0 = 0, \quad (1.1)$$

where $U$ is a horizontal lift of $X$ taking values in the frame bundle on $M$ and $p$ is the canonical projection in $TM$ of a vertical vector of $TTM$. The map $A$ will denote the antidevelopment with respect to $\nabla$ and $A'$ the antidevelopment with respect to $\nabla'$.

The initial motivation of this paper was to answer the following question: For some open interval $I$ in $\mathbb{R}$, consider a family $(X_t(a))_{a \in I, t \in [0,\xi(a)]}$ of continuous martingales $X(a)$ in $M$, each with lifetime $\xi(a)$, differentiable in $a$ for the topology of compact convergence in probability. Is then also $(X(a), A(X(a)))$ differentiable in $a$, and if the answer is positive, do we have the relation $s(\partial_a A(X(a))) = A'(\partial_a X(a))$ (where $\partial_a$ denotes differentiation with respect to $a$ and $s$ is the map $TTM \rightarrow TTM$ defined by $s(\partial_a \partial_t x(t,a)) = \partial_t \partial_a x(t,a)$, if $(t,a) \mapsto x(t,a)$ is smooth and takes its values in $M$)?

A positive answer will be given to this question, and this result will be obtained as a particular case of general theorems on stability of stochastic differential equations.

In this paper equations of the general type

$$DV(a) = f(X(a), Z(a)) DX(a) \quad (1.2)$$

between two manifolds $M$ and $N$ are studied, where $DX(a)$ denotes the (formal) differential of order 2 of $X(a)$, and $f$ is a Schwartz morphism between the second order bundles $\tau M$ and $\tau N$. The topology of semimartingales, defined in [E1] for $\mathbb{R}$-valued processes, will be adapted to manifold-valued semimartingales with lifetime. In particular, it will be shown that the map $(X, f, Z_0) \mapsto (X, Z)$ is continuous, where $Z$ is the maximal solution starting from $Z_0$ to $DV = f(X, Z)DX$, with appropriate topologies on both sides.

When applied to a certain family of semimartingales and an appropriate Schwartz morphism, this result will tell us that if $a \mapsto X(a)$ is $C^1$ in the topology of semimartingales, and further if $f$ is $C^1$ with locally Lipschitz derivative, $Z(a)$ the maximal solution to (1.2) with $(Z_0(a))_{a \in I} C^1$ in probability, then $a \mapsto (X(a), Z(a))$ is $C^1$ in the topology of semimartingales and the derivative $\partial_a Z(a)$ is the maximal solution to

$$DV \partial_a Z(a) = f'(\partial_a X(a), \partial_a Z(a)) DV \partial_a X(a) \quad (1.3)$$

where $f'$ is a Schwartz morphism between the second order bundles $\tau TM$ and $\tau TN$.

As a corollary, we obtain results on differentiability of solutions to Stratonovich and Itô equations. It will be shown that they can be differentiated in the same way as solutions to ordinary differential equations (for the Itô case, the Itô differentials of the derivative process have to be defined with the complete lifts of the connections).

If $M$ is endowed with a connection $\nabla$, then it will be shown that, as in the flat case, the topology of semimartingales and the topology of uniform convergence in probability on compact sets coincide on the set of martingales. Using these results it will be possible to prove commutativity of antidevelopment and differentiation.

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