CHAPTER IV

OPERATIONS WITH s-MANIFOLDS

Submanifolds and foliations.

Let \((M, \nabla)\) be a manifold with an affine connection. A submanifold \(N \subset M\) is said to be autoparallel if for every curve \(\gamma: \langle 0, 1 \rangle \to N\) and every tangent vector \(u \in T_{\gamma(0)} N\) the parallel displacement of \(u\) along \(\gamma\) yields a tangent vector to \(N\), i.e., \(h^\gamma u \in T_{\gamma(1)} N\) (cf. [KN II]).

If \(N \subset M\) is autoparallel, then the connection \(\nabla\) on \(M\) naturally induces a connection \(\nabla^N\) on \(N\) as follows: for \(p \in N\), \(u \in T_p N\), \(Y \in \mathcal{X}(N)\) we put \(\nabla^N_u Y = \nabla_u Y\), where \(Y\) is any extension of \(Y\) from a neighborhood of \(p\) in \(N\) to a neighborhood of \(p\) in \(M\).

**Definition IV.1.** Let \((M, \{s_x^1\})\) be a regular s-manifold. A submanifold \(N \subset M\) is said to be invariant if the following holds: for every two points \(p, q \in N\) and every local automorphism \(\varphi\) of \(M\) such that \(\varphi(p) = q\) we have \(\varphi(N \cap \varphi^{-1}(M)) \subset N\).

**Proposition IV.1.** Every invariant submanifold \(N \subset M\) is autoparallel with respect to the canonical connection \(\nabla\) of \((M, \{s_x^1\})\).

**Proof.** Let \(N \subset M\) be invariant under all local automorphisms. Let \(\gamma: \langle 0, 1 \rangle \to N\) be a curve and \(h^\gamma: M_{\gamma(0)} \to M_{\gamma(1)}\) the parallel displacement along \(\gamma\). Then there is a transvection \(g \in \text{Tr}(M, \{s_x^1\})\) such that \(g(\gamma(0)) = \gamma(1)\) and \(g_{\gamma} = h^\gamma\) on \(M_{\gamma(0)}\). (Cf. Theorem II.32.) Now, because \(N\) is invariant, we get \(g_{\gamma}^*(\gamma(0)) = N_{\gamma(1)}\) and hence \(h^N_{\gamma(0)} = N_{\gamma(1)}\).

**Proposition IV.2.** Let \(N \subset M\) be an invariant submanifold of \((M, \{s_x^1\})\). Then \(N\) is naturally a regular s-manifold \((N, \{s_y^1|_N\})\) and its canonical connection coincides with the induced connection \(\nabla^N\).

**Proof.** Obviously, \(\nabla^N\) is invariant under all \(s_y|_N\), \(y \in N\), and \(\nabla^N(S|_N) = 0\). Now, we use the uniqueness part of Theorem II.4, (A).
(N,\{s\}_{y}|_{\mathcal{N}}) is called an invariant submanifold of \((M,\{s\}_{x})\).

Let \((V,S,\tilde{R},\tilde{T})\) be an infinitesimal s-manifold. An infinitesimal s-manifold \((W,S',\tilde{R}',\tilde{T}')\) is called a submanifold of \((V,S,\tilde{R},\tilde{T})\) if \(W \subseteq V\), \(S' = S|_{W}\), \(\tilde{R}' = \tilde{R}|_{W}\), \(\tilde{T}' = \tilde{T}|_{W}\). Let us remark that each submanifold of \((V,S,\tilde{R},\tilde{T})\) is uniquely defined by a subspace \(W \subseteq V\) such that \(S(W) \subseteq W\), \(\tilde{R}(W,W) \subseteq W\), \(\tilde{T}(W,W) \subseteq W\). Further, a subspace \(W \subseteq V\) (or, a submanifold \((W,S',\tilde{R}',\tilde{T}')\) of \((V,S,\tilde{R},\tilde{T})\) respectively) is said to be invariant if it is invariant with respect to all automorphisms of \((V,S,\tilde{R},\tilde{T})\).

**Proposition IV.4.** Let \((M,\{s\}_{x})\) be a connected regular s-manifold and \((V,S,\tilde{R},\tilde{T})\) its infinitesimal model. If \(N,\{s\}_{y}|_{\mathcal{N}}\) is an invariant submanifold of \((M,\{s\}_{x})\) then its infinitesimal model is canonically an invariant submanifold \((W,S',\tilde{R}',\tilde{T}')\) of \((V,S,\tilde{R},\tilde{T})\).

**Proof.** First, let us choose \(p \in N \subseteq M\) and put \(V = T_p(M)\), \(W = T_p(N)\). From Proposition IV.3 we obtain that \((W,S',\tilde{R}',\tilde{T}')\) is a submanifold of \((V,S_p,\tilde{R}_p,\tilde{T}_p)\), and the invariance of \(W \subseteq V\) follows from the invariance of \(N \subseteq M\) at the point \(p\). Our formulation in terms of the infinitesimal models is now justified by the invariance of \(W \subseteq V\).

**Theorem IV.5.** Let \((M,\{s\}_{x})\) be a connected regular s-manifold and \((V,S,\tilde{R},\tilde{T})\) its infinitesimal model. Let \((W,S',\tilde{R}',\tilde{T}')\) be an m-dimensional invariant submanifold of \((V,S,\tilde{R},\tilde{T})\). Then there is an m-dimensional involutive distribution \(\Delta\) on \(M\) the maximal integral manifolds of which are invariant submanifolds of \((M,\{s\}_{x})\) with the infinitesimal model \((W,S',\tilde{R}',\tilde{T}')\).

**Proof.** Let us define the distribution \(\Delta\) as follows: for each \(p \in M\) choose an isomorphism \(f: (V,S,\tilde{R},\tilde{T}) \rightarrow (M_p,\{s\}_{x_p})\) and put \(\Delta_p = f(W)\). Then \(\Delta\) is invariant with respect to all local automorphisms of \((M,\{s\}_{x})\). Now, the parallel displacement along \(\gamma = pq\) with respect to \(\nabla\) is induced by a transvection \(\varphi\) such that \(\varphi(p) = q\). Because every transvection is an automorphism, \(\varphi_*\) sends \(\Delta_p\) into \(\Delta_q\), and hence \(\Delta_q\) is the effect of the parallel transport of \(\Delta_p\) along \(\gamma\). We conclude that the distribution \(\Delta\) is parallel with respect to the connection \(\nabla\). Let now \(X, Y\) be two local vector fields on \(M\) belonging to the distribution \(\Delta\). Then \([X,Y] = \nabla_X Y - \nabla_Y X - \tilde{T}(X,Y)\).