1. Preliminary remarks

One of the motivating subjects for my lecture is the classical Dirichlet problem for elliptic equations. For that reason it is useful to start with some concise remarks describing the situation in the case of the Laplace equation.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $g \in C(\partial \Omega)$ a continuous function on the boundary $\partial \Omega$. Concerning the Dirichlet problem there is a decomposition $\partial \Omega = \partial_r \Omega \cup \partial_i \Omega$ of the boundary $\partial \Omega$ ($\partial_r \Omega$ regular, $\partial_i \Omega$ irregular boundary points) and an unique bounded harmonic function $u_g$ in $\Omega$ with

$$\lim_{x \to y} u_g(x) = g(y) \text{ for every } y \in \partial_r \Omega \text{ and every } g \in C(\partial \Omega).$$

Moreover, there exists a family $(\mu_x)_{x \in \Omega}$ of harmonic measures, such that

$$u_g(x) = \int g(y) \, d\mu_x(y).$$

The function $u_g$ is the so-called generalized solution of the Dirichlet problem (see e.g. L. L. Helms /18/). N. Wiener /25/ constructed this solution in the following way. Let $(\Omega_k)_{k=1,2,...}$ be a sequence of smooth domains with $\Omega_k \subset \Omega_{k+1} \subset \Omega$ ($k=1,2,...$), $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$, $f \in C(\mathbb{R}^n)$ an extension of $g \in C(\partial \Omega)$ ($u|_{\partial \Omega} = g$) and $u_{k,f}$ the sequence of the solutions of the Dirichlet problems

$$\Delta u_{k,f} = 0 \text{ in } \Omega_k, \quad u_{k,f}|_{\partial \Omega_k} = f.$$  

Then the generalized solution $u_g$ is the uniform limit of $u_{k,f}$ in $\Omega$ (i.e., $u_g$ is independent of the approximation of the domain and of the special extension $f$).
2. The Dirichlet problem for higher order elliptic equations

In the case of elliptic equations of higher order we do not have such definitive results as in the case of equations of second order. In the following we shall try to characterize the actual situation. Let

\[ L(x,D) := \sum_{|\alpha| \leq 2m} a_{\alpha}(x)D^{\alpha} \]

be a properly elliptic differential operator with smooth coefficients. A continuous vector function \( \tilde{g} = (g_\alpha)_{|\alpha| \leq 1} \) on a compact set \( K \subset \mathbb{R}^n \) is called a Whitney-Taylorfield of order 1, if there exists a function \( g \in C^1(\mathbb{R}^n) \) such that \( D^{\alpha}g |_{K} = g_\alpha \ (|\alpha| \leq 1) \). Let \( W^1(K) \) be the vector space of all this Whitney-Taylorfields.

\[ \| \tilde{g} \| = \sum_{|\alpha| \leq 1} \| g_\alpha \|_{C(K)} \]

is a norm in \( W^1(K) \).

Using \( W^{m-1}(\partial\Omega) \) we are able to formulate the Dirichlet problem in the following natural way.

**Dirichlet problem (classical formulation):**

Let \( \Omega \subset \mathbb{R}^n \) be a given bounded domain and \( \tilde{g} \in W^{m-1}(\partial\Omega) \). We are looking for a function \( u \in C^2(\Omega) \cap C^{m-1}(\bar{\Omega}) \) with \( Lu = 0 \) in \( \Omega \) and \( D^{\alpha}u |_{\partial\Omega} = g_\alpha \ (|\alpha| \leq m-1) \).

In the case \( \text{ord } L > 2 \) this problem is far from a definitive solution. Usually the Dirichlet problem is studied in the so-called weak form.

**Dirichlet problem (weak formulation):**

For given \( g \in W^m_2(\Omega) \) we are looking for a weak solution of the equation \( Lu = 0 \) (i.e. \( \int u L^*\varphi \, dx = 0 \) for every \( \varphi \in C^\infty_0(\Omega) \)) with \( u-g \in W^m_2(\Omega) \).

Here \( W^m_2(\Omega) \) denotes the classical Sobolev space, \( W^m_2(\Omega) \) the closure of \( C^\infty_0(\Omega) \) in \( W^m_2(\Omega) \) and \( L^* \) the adjoint differential operator.

For a sufficiently smooth boundary \( \partial\Omega \) the solution of the weak problem is a classical solution (see e.g. S. Agmon, A. Douglis, L. Nirenberg /3/), but also for non-smooth domains a lot of results...